Bounding Least Common Multiples with Triangles

ITP 2016: Proof Pearl

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College of Engineering and Computer Science
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AKS mechanisation

PRIMES is in P

Manindra Agrawal Neeraj Kayal
Nitin Saxena*

Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

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We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

We will need the following simple fact about the lcm of first m numbers (see, e.g., [Nai82] for a proof).

Lemma 3.1. *Let $LCM(m)$ denote the lcm of first m numbers. For $m \geq 7$:*

$$LCM(m) \geq 2^m.$$

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Lemma 3.1. *Let $LCM(m)$ denote the lcm of first m numbers. For $m \geq 7$:*

$$LCM(m) \geq 2^m.$$

- ◦ Replacing 2^m by $2^m/2$ makes the lower bound valid for all $m > 0$.
- ◦ This change won't affect the conclusion: AKS algorithm is in class P.

Nair's Paper

ON CHEBYSHEV-TYPE INEQUALITIES FOR PRIMES

M. NAIR
Department of Mathematics, University of Glasgow, Glasgow, Scotland

Proof. For $1 \leqslant m \leqslant n$, consider the integral

$$I = I(m,n) = \int_0^1 x^{m-1}(1-x)^{n-m}\,dx = \sum_{r=0}^{n-m} (-1)^r \binom{n-m}{r} \frac{1}{m+r}. \tag{7}$$

Clearly, $\mathrm{Id}_n \in \mathbb{N}$. On the other hand, repeated integration by parts yields

$$I = 1/m\binom{n}{m}. \tag{8}$$

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In conclusion, it is perhaps appropriate to point out that Theorem 3 can also be proved by the standard methods of proof. The interest here lies essentially in the rather curious nature of this proof. It is unexpected to use (i) to prove (ii), and it certainly is strange that there is no mention of primes in the proof of Theorem 3. It also seems worthwhile to point out that there are different ways to prove the identity implied by equations (7) and (8), for example, by expressing $1/x(x+1)\cdots(x+m)$ in partial fractions or by using the difference operator.

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◦ The cryptic "difference operator" means Leibniz's Harmonic Triangle!

LCM of a List

list_lcm [1]
list_lcm [1; 2]
list_lcm [1; 2; 3]
list_lcm [1; 2; 3; 4]
list_lcm [1; 2; 3; 4; 5]
list_lcm [1; 2; 3; 4; 5; 6]

LCM of a List

$\texttt{list\_lcm}\,[1] = 1$
$\texttt{list\_lcm}\,[1;2] = 2$
$\texttt{list\_lcm}\,[1;2;3] = 6$
$\texttt{list\_lcm}\,[1;2;3;4] = 12$
$\texttt{list\_lcm}\,[1;2;3;4;5] = 60$
$\texttt{list\_lcm}\,[1;2;3;4;5;6] = 60$

LCM of a List

$$
\begin{aligned}
\texttt{list\_lcm}\,[1] &= 1 & \geq 2^0 = 1\\
\texttt{list\_lcm}\,[1;2] &= 2 & \geq 2^1 = 2\\
\texttt{list\_lcm}\,[1;2;3] &= 6 & \geq 2^2 = 4\\
\texttt{list\_lcm}\,[1;2;3;4] &= 12 & \geq 2^3 = 8\\
\texttt{list\_lcm}\,[1;2;3;4;5] &= 60 & \geq 2^4 = 16\\
\texttt{list\_lcm}\,[1;2;3;4;5;6] &= 60 & \geq 2^5 = 32
\end{aligned}
$$

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Theorem

Lower bound for the LCM of consecutive numbers.

$\vdash 2^n \leq \texttt{list\_lcm}\ [1\ ..\ n + 1]$

LCM Lower Bound

Let ℓ = `[`*a*`;` *b*`;` *c*`]`.

LCM Lower Bound

Let ℓ = [a; b; c].
Since *LCM* is a common multiple of each element (in fact, the least),

$$a \leq \texttt{list\_lcm}\ [a;\ b;\ c]$$
$$b \leq \texttt{list\_lcm}\ [a;\ b;\ c]$$
$$c \leq \texttt{list\_lcm}\ [a;\ b;\ c]$$

LCM Lower Bound

Let ℓ = [*a*; *b*; *c*].
Since *LCM* is a common multiple of each element (in fact, the least),

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$$b \leq \texttt{list\_lcm [}a\texttt{; }b\texttt{; }c\texttt{]}$$
$$c \leq \texttt{list\_lcm [}a\texttt{; }b\texttt{; }c\texttt{]}$$

Hence

$$a + b + c \leq 3 \times \texttt{list\_lcm [}a\texttt{; }b\texttt{; }c\texttt{]}$$

LCM Lower Bound

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$$c \le \texttt{list\_lcm}\ [a;\ b;\ c]$$

Hence

$$a + b + c \le 3 \times \texttt{list\_lcm}\ [a;\ b;\ c]$$

Theorem

For a list ℓ of positive numbers, SUM $\ell \le$ LENGTH $\ell \times$ list_lcm ℓ.

LCM Lower Bound – Applications

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Naïve application:

$$\frac{(n+1)(n+2)}{2} \leq (n + 1) \times \texttt{list\_lcm}\ [1\ ..\ n + 1]$$

$$\frac{(n+2)}{2} \leq \texttt{list\_lcm}\ [1\ ..\ n + 1]$$

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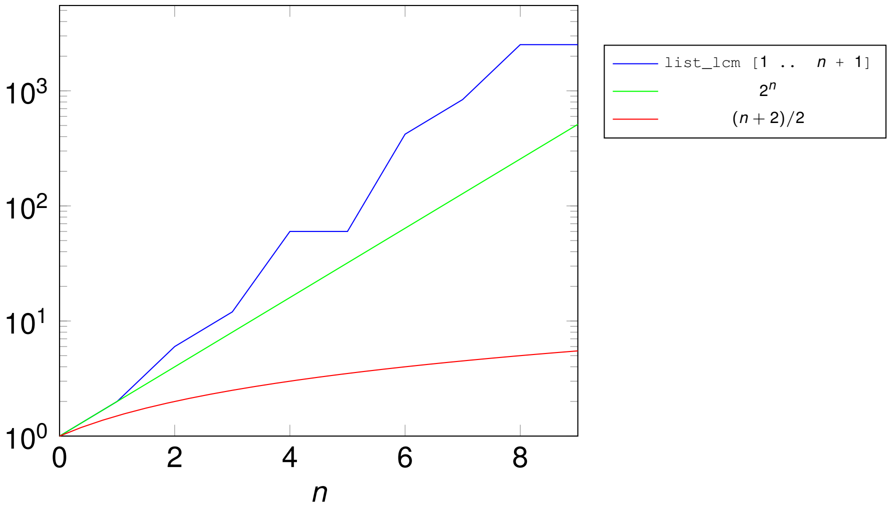
$$\frac{(n+2)}{2} \leq \texttt{list\_lcm}\ [1\ ..\ n + 1]$$

Need a clever idea to obtain this lower bound:

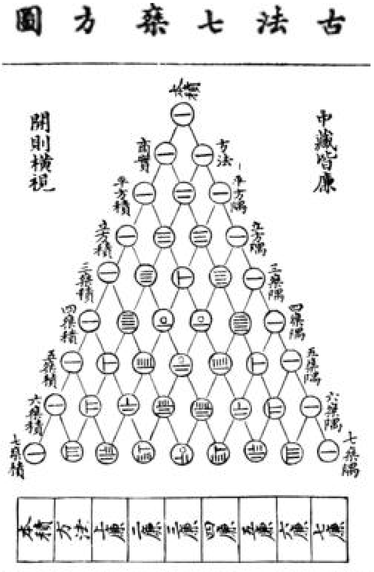
much better!

$$2^n \leq \texttt{list\_lcm}\ [1\ ..\ n + 1]$$

LCM Bound Comparison



Yang Hui's Triangle



Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

- Each boundary entry: always 1.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
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- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

Pascal's Triangle

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1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

Pascal's Triangle

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

Sum of the n-th row:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Leibniz's Harmonic Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \frac{1}{1} & & & & & & \\
 & & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & & \\
 & & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & & \\
 & & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & & \\
 & \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} & \\
\frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7}
\end{array}
$$

Leibniz's Harmonic Triangle

$$\frac{1}{1}$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$
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- Each boundary entry: $\dfrac{1}{(n+1)}$ for the n-th row, $n = 0, 1, \cdots$

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- Each entry (inside or not): sum of two immediate children.

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\end{array}
$$

- Each boundary entry: $\dfrac{1}{(n+1)}$ for the n-th row, $n = 0, 1, \cdots$
- Each entry (inside or not): sum of two immediate children.

Formula for k-th entry at n-th row: $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{1}{(n+1)\binom{n}{k}}$, $k = 0, 1, \cdots, n$.

Leibniz's Denominator Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

$$\mathcal{L}_{nk} = (n + 1) \times \binom{n}{k}, \qquad k = 0, 1, \cdots, n.$$

Leibniz's Denominator Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

- Each left boundary entry: $(n+1)$ for the n-th row, $n = 0, 1, \cdots$

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- Other entries: need a counter-part of the sum-of-children rule;

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  i.e., a relationship for L-shaped triplets.

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	2	2				
	3	6	3			
	4	12	12	4		
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$6\times$	1	5	10	10	5	1

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$$\mathcal{L}_{nk} = (n + 1) \times \binom{n}{k}, \qquad k = 0, 1, \cdots, n.$$

By $\mathcal{L}_{nk}$, the horizontal row is just a multiple of the binomial row.

Clever Idea: a sneak preview

	1					
	2	2				
	3	6	3			
	4	12	12	4		
	5	20	30	20	5	
	6	30	60	60	30	6
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Theorem (Lower Bound for the LCM of any list ℓ)

For a list ℓ of positive numbers, $\texttt{SUM}\ \ell \leq \texttt{LENGTH}\ \ell \times \texttt{list\_lcm}\ \ell$.

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- Applying theorem to vertical list . . . a disappointing lower bound.

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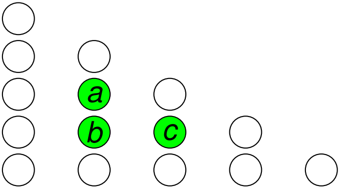
- Applying theorem to vertical list . . . a disappointing lower bound.
- Applying theorem to horizontal list . . . ??
  Only if both lists have the same LCM . . . will this hold?

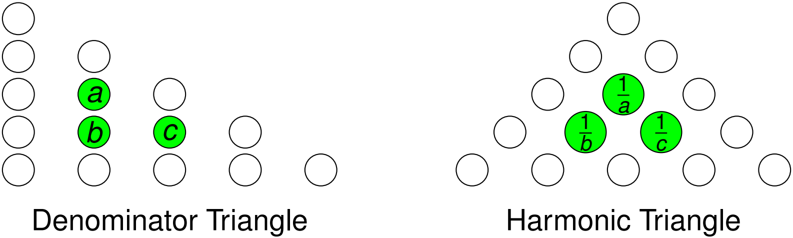
Leibniz Triplet



Denominator Triangle

- Each left boundary entry: $(n + 1)$ for the n-th row, $n = 0, 1, \cdots$

Leibniz Triplet



By sum-of-children rule:

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c},$$

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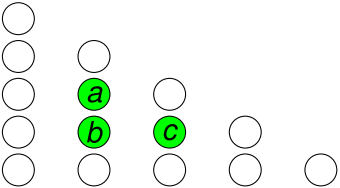
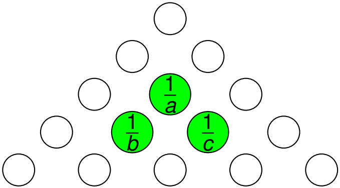


Denominator Triangle



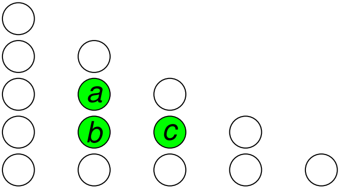
Harmonic Triangle

By sum-of-children rule:

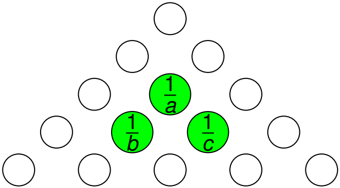
$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}, \quad \text{or} \quad \frac{1}{c} = \frac{1}{a} - \frac{1}{b}$$ "difference operator"

- Each left boundary entry: $(n+1)$ for the n-th row, $n = 0, 1, \cdots$

Leibniz Triplet



Denominator Triangle



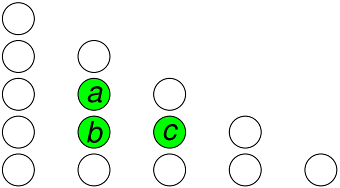
Harmonic Triangle

By sum-of-children rule:

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}, \quad \text{or} \quad \frac{1}{c} = \frac{1}{a} - \frac{1}{b}$$ "difference operator"

- Each left boundary entry: $(n+1)$ for the n-th row, $n = 0, 1, \cdots$
- Each Leibniz triplet: $c = \dfrac{ab}{(b-a)}$,

Leibniz Triplet



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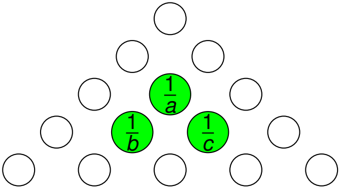

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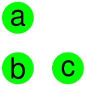
- Each left boundary entry: $(n + 1)$ for the n-th row, $n = 0, 1, \cdots$
- Each Leibniz triplet: $c = \dfrac{ab}{(b - a)}$, or $ab = c(b - a)$.

Leibniz Triplet Property

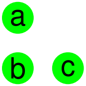


For a Leibniz triplet $\{a, b, c\}$, $\quad ab = c(b - a)$.

Leibniz Triplet Property

Theorem (LCM Exchange)

For a Leibniz triplet $\{a, b, c\}$*,* `lcm` b c = `lcm` b a*.*



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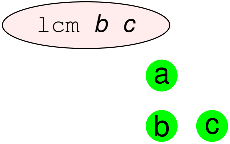

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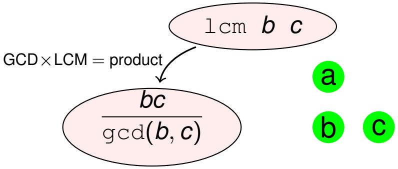

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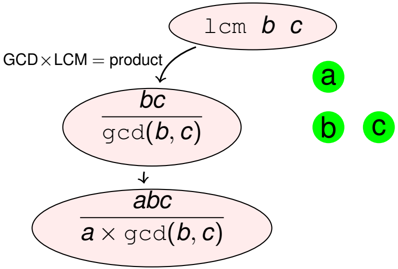

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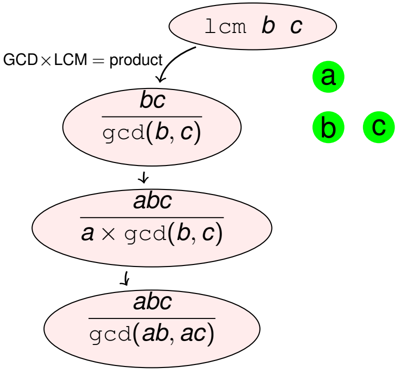

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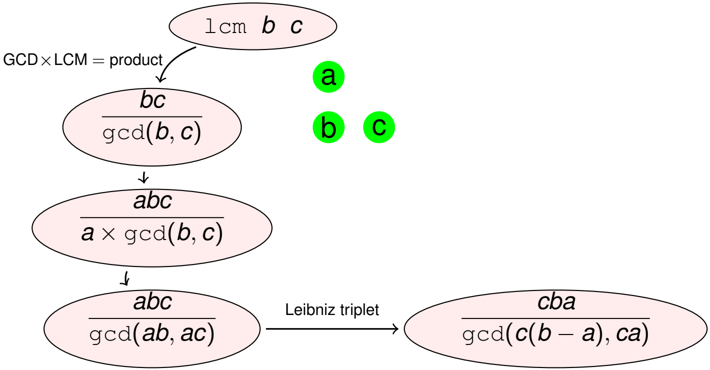

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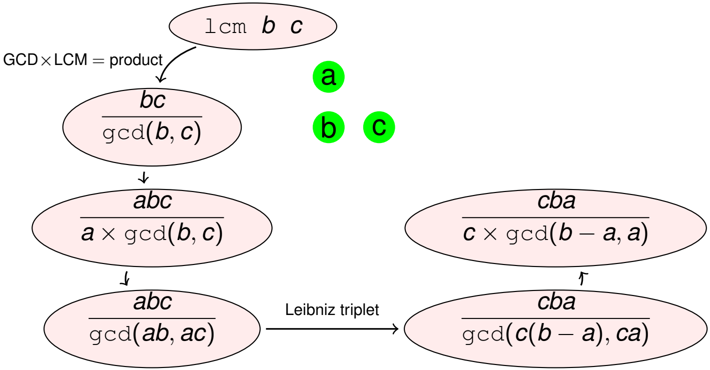

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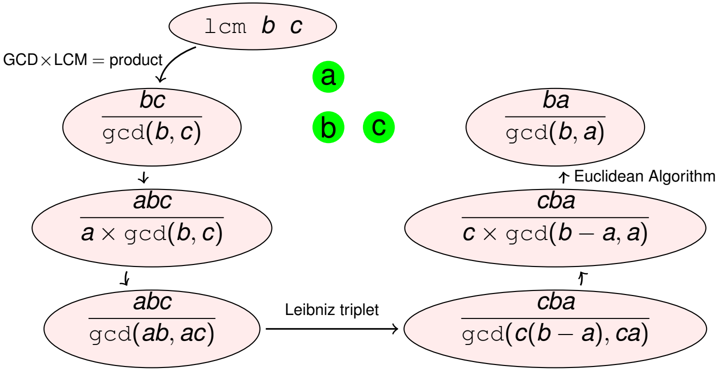

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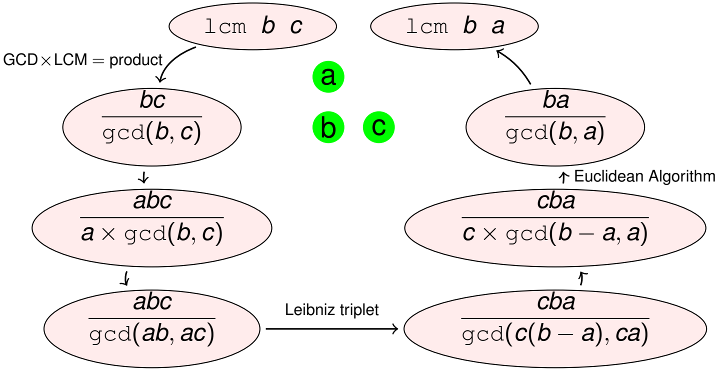

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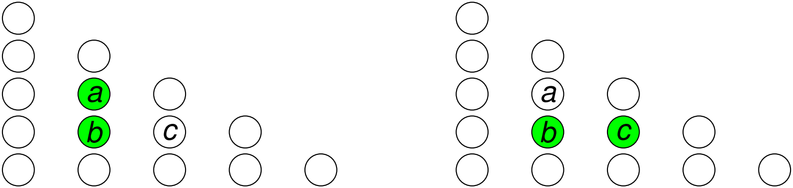
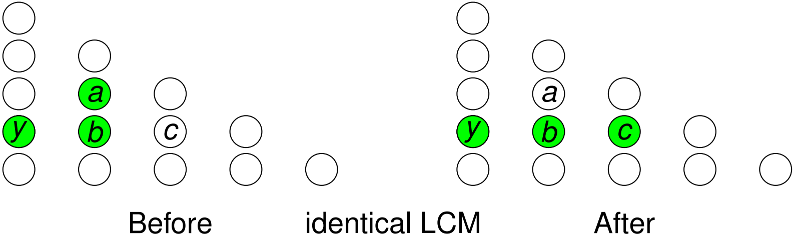
Zig-zag Paths



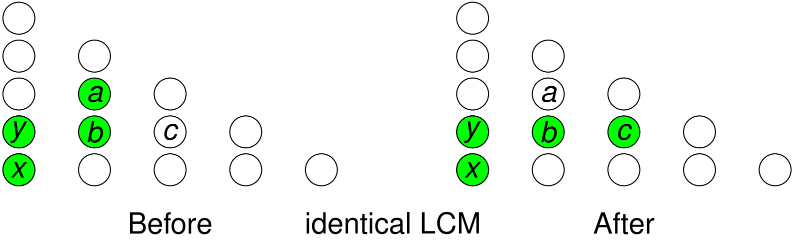
Before identical LCM After

Zig-zag Paths



- Arms of a Leibniz triplet extend to paths, keeping overall LCM.

Zig-zag Paths



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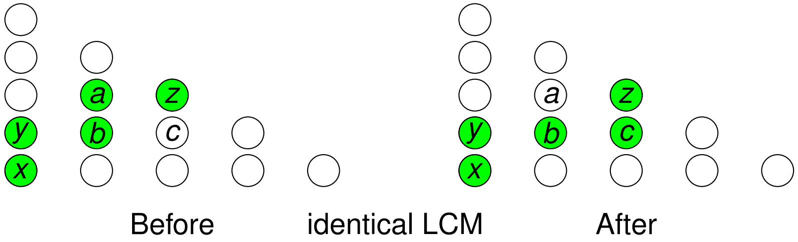
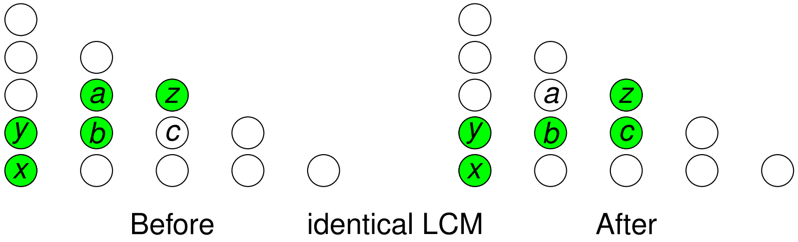


- Arms of a Leibniz triplet extend to paths, keeping overall LCM.
- A path can zig-zag to another by a suitable Leibniz triplet.

Zig-zag Paths



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Theorem

$\vdash p_1 \rightsquigarrow p_2 \Rightarrow \texttt{list\_lcm}\ p_1 = \texttt{list\_lcm}\ p_2$

Wriggle Paths

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

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- Transform a path by successive zig-zags keeps overall LCM.

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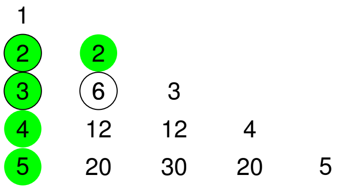

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Wriggle Paths

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2	**2**			
3	**6**	3		
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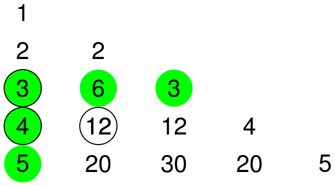

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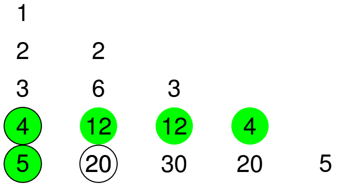

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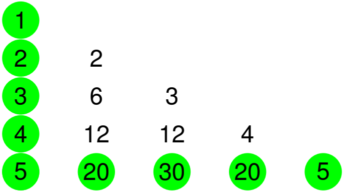

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Proof Idea

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2	2			
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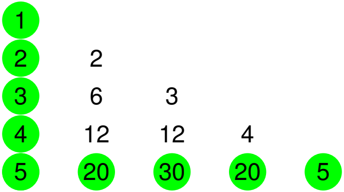
`list_lcm [1; 2; 3; 4; 5]`

Proof Idea




```
   list_lcm [1; 2; 3; 4; 5]
=  list_lcm [5; 20; 30; 20; 5]
```
by wriggling path transform

Proof Idea

1				
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3	6	3		
4	12	12	4	
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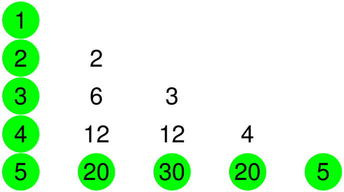
```
  list_lcm [1; 2; 3; 4; 5]
= list_lcm [5; 20; 30; 20; 5]          by wriggling path transform
= 5 × list_lcm [1; 4; 6; 4; 1]         note 5 = LENGTH of list
```

Proof Idea



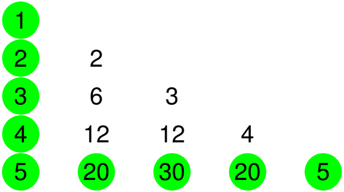
$\quad$ `list_lcm [1; 2; 3; 4; 5]`

$=$ `list_lcm [5; 20; 30; 20; 5]` — by wriggling path transform

$=$ $5 \times$ `list_lcm [1; 4; 6; 4; 1]` — note $5 =$ `LENGTH` of list

$\geq$ $1 + 4 + 6 + 4 + 1$ — by `SUM` $\ell \leq$ `LENGTH` $\ell \times$ `list_lcm` ℓ

Proof Idea

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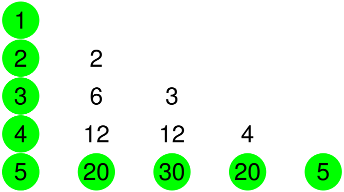
   list_lcm [1; 2; 3; 4; 5]
=  list_lcm [5; 20; 30; 20; 5]          by wriggling path transform
=  5 × list_lcm [1; 4; 6; 4; 1]         note 5 = LENGTH of list
≥  1 + 4 + 6 + 4 + 1                    by SUM ℓ ≤ LENGTH ℓ × list_lcm ℓ
=  (1 + 1)^4 = 2^4                      by binomial expansion.
```

Proof Idea



list_lcm [1; 2; 3; 4; 5]

= list_lcm [5; 20; 30; 20; 5] — by wriggling path transform

= 5 × list_lcm [1; 4; 6; 4; 1] — note 5 = LENGTH of list

= list_lcm [1; 4; 6; 4; 1] — by unrolling multiplication

+ list_lcm [1; 4; 6; 4; 1]

+ list_lcm [1; 4; 6; 4; 1]

+ list_lcm [1; 4; 6; 4; 1]

+ list_lcm [1; 4; 6; 4; 1] — picking diagonal elements

≥ 1 + 4 + 6 + 4 + 1 — by SUM ℓ ≤ LENGTH ℓ × list_lcm ℓ

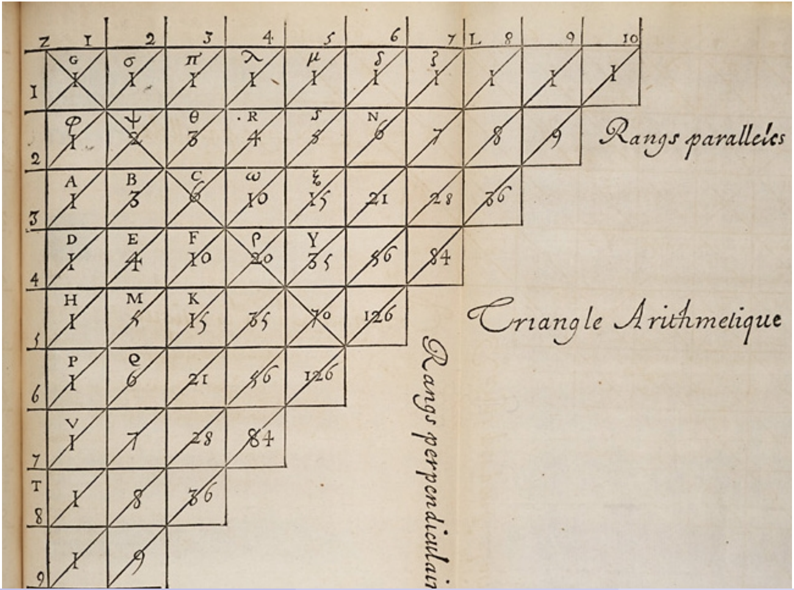
= $(1 + 1)^4 = 2^4$ — by binomial expansion.

Reference

Questions

- Scripts
  `https://bitbucket.org/jhlchan/hol/src/`
  subfolder: `algebra/lib`.
- Paper
  `https://bitbucket.org/jhlchan/hol/downloads`

Pascal's Triangle (1665)



Leibniz's Triangle (1672)

1	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20
1	5	15	35
1	6	21	56
1	7	28	84

	0	1	2	3	4	5	...
	$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	...
	$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...
	$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	...
	$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	...
	⋮	⋮	⋮	⋮	⋮	⋮	...
sum	$\frac{0}{0}$	$\frac{1}{0}$	$\frac{1}{0}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{4}{3}$	...

To see the modern Leibniz's Triangle, read the top: skip columns "0" and "1", divide column "2" by 1, column "3" by 2, column "4" by 3, *etc.*

Math Stack Exchange

Is there a direct proof of this lcm identity?

26

The identity

$$(n+1)\text{lcm}\left(\binom{n}{0},\binom{n}{1},\ldots\binom{n}{n}\right) = \text{lcm}(1,2,\ldots n+1)$$

is probably not well-known. The only way I know how to prove it is by using Kummer's theorem that the power of p dividing $\binom{a+b}{a}$ is the number of carries needed to add a and b in base p. Is there a more direct proof, e.g. by showing that each side divides the other?

7

(number-theory) (binomial-coefficients)

share cite improve this question | edited Aug 3 '10 at 8:04 | asked Aug 3 '10 at 4:18

3 Answers

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19

Consider *Leibniz harmonic triangle* — a table that is like «Pascal triangle reversed»: on it's sides lie numbers $\frac{1}{n}$ and each number is the sum of two beneath it (see the picture).

One can easily proove by induction that m-th number in n-th row of Leibniz triangle is $\frac{1}{(n+1)\binom{n}{m}}$.