The Flow of ODEs

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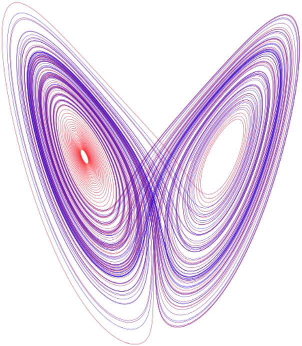
Technische Universität München

Introduction

Motivation

- Lorenz attractor, chaos



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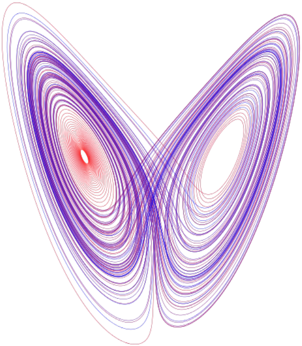

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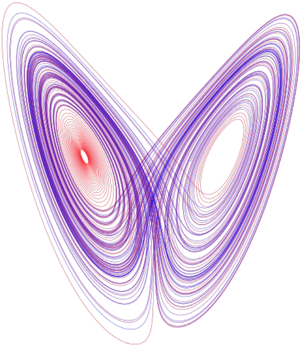

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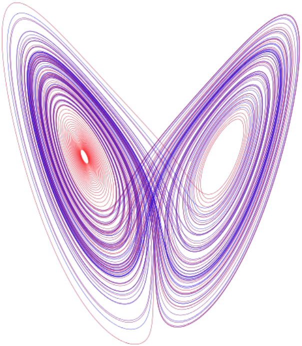

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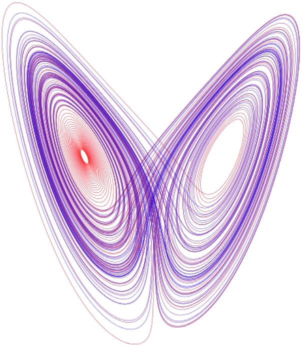

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Contribution

- formalization of *flow*:
  general theory for dependence on initial conditions
- use existing verified ODE-solver [Immler, TACAS 2015]:
  bounds on variational equation

Structure

Flow

Dependence on Initial Condition

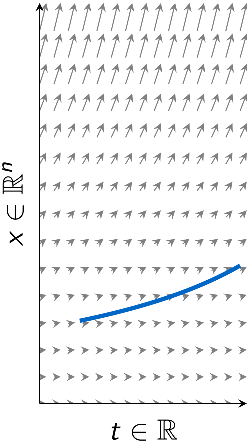
Numerics

Structure

The Flow of ODEs

- ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t))$$



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- [Immler, Hölzl @ ITP 2012]: initial value problems

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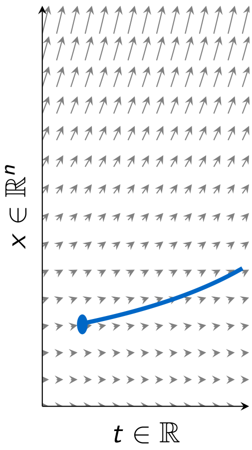

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- formalize flow $\varphi(x_0, t)$: solution w.r.t. initial condition

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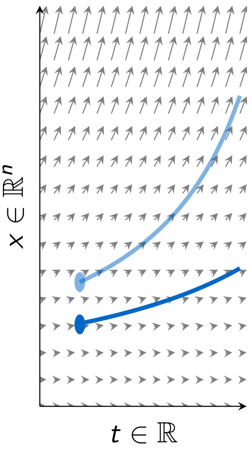

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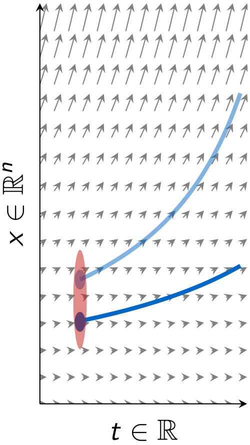

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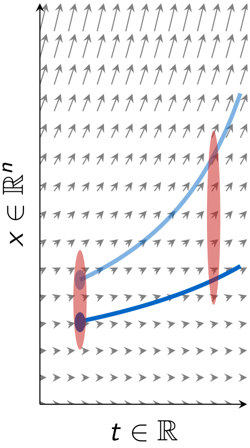

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  - quantitative: differentiable

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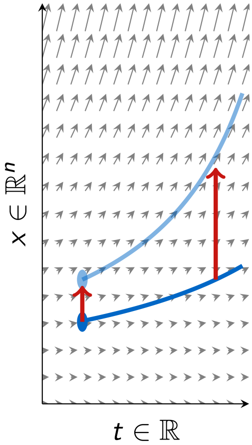

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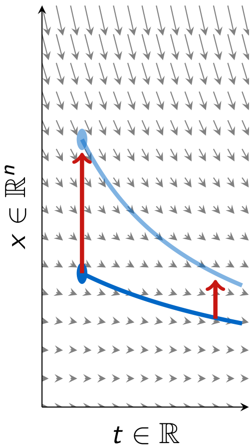
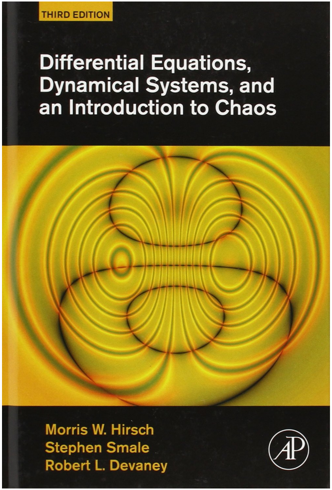


Formalization

- continuity and differentiability are "natural" properties (chapter 7):



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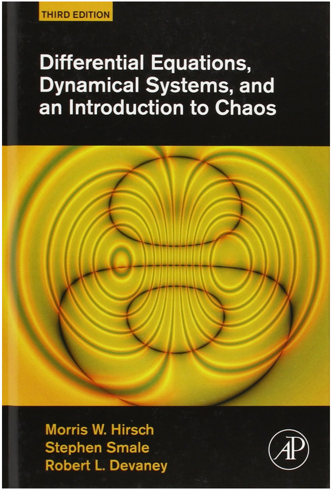

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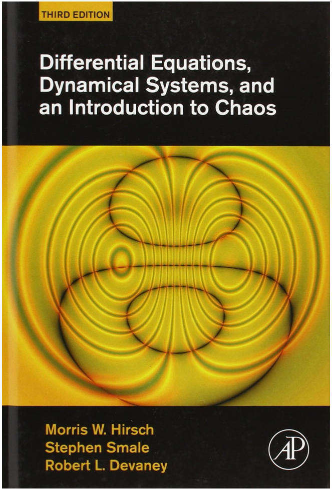

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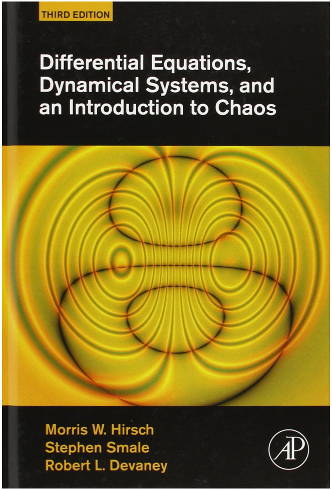

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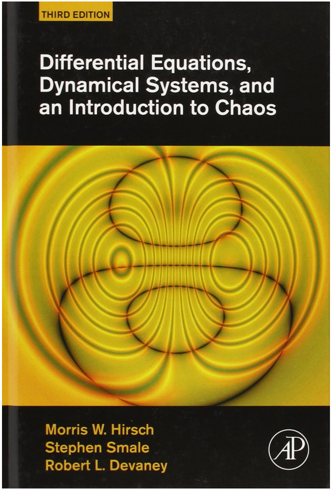

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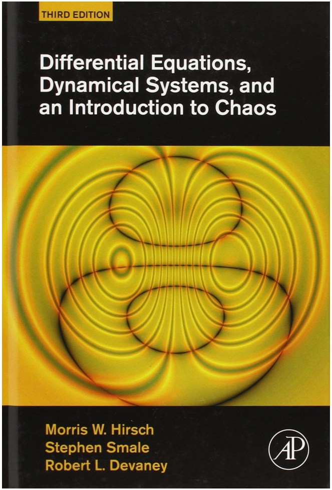
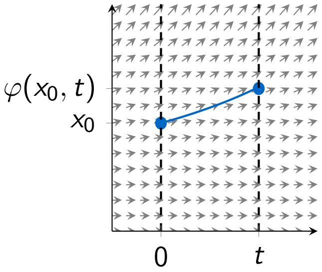
  - proofs in chapter 17
- interface to the rest of the theory that hides technical constructions



The Interface: *ex-ivl* and φ

- locally Lipschitz continuous
  $f : \mathbb{R}^n \to \mathbb{R}^n$ (on open set X)

$\dot{x}(t) = f(x(t))$



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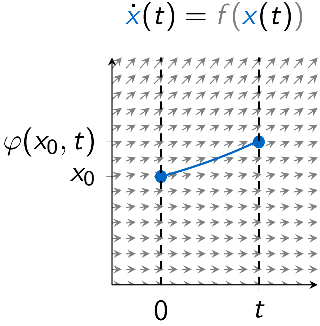

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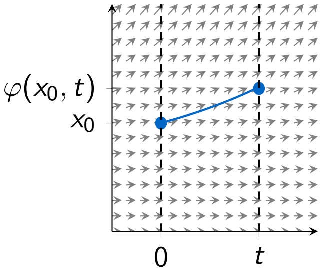

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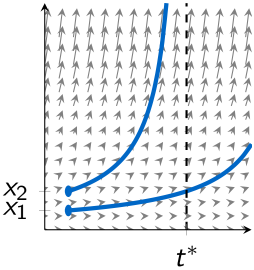

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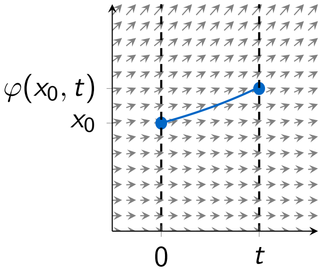

Theorem (flow solves IVP)

For $t \in \textit{ex-ivl}(x_0)$:

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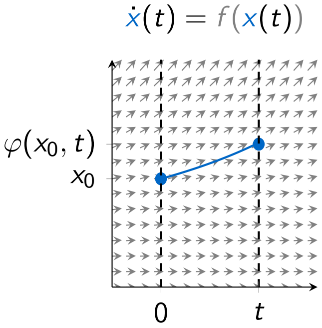

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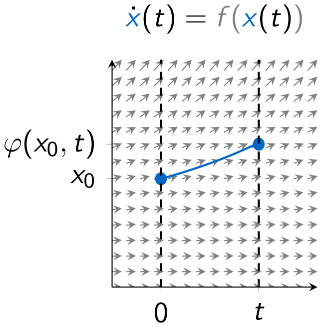

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For $t \in \textit{ex-ivl}(x_0)$:

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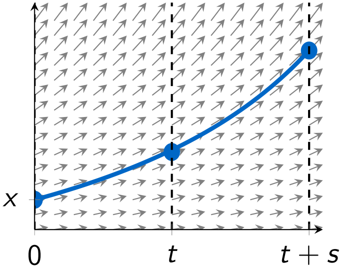
- $\varphi(x_0, 0) = x_0$

Flow property



Theorem (Flow property)

$(t \in \textit{ex-ivl}(x) \wedge s \in \textit{ex-ivl}(\varphi(x, t))) \implies$
$\varphi(x, t + s) = \varphi(\varphi(x, t), s)$

Structure

Flow

Dependence on Initial Condition

Numerics

Structure

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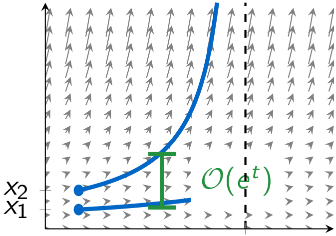
Technical Lemmas

- Grönwall lemma

$$\textit{continuous-on}\ [0;a]\ g \implies$$
$$\forall t.\ 0 \le g(t) \le C + K \cdot \int_0^t g(s)\,\mathrm{d}s \implies$$
$$\forall t \in [0;a].\ g(t) \le C \cdot e^{K \cdot t}$$

Technical Lemmas

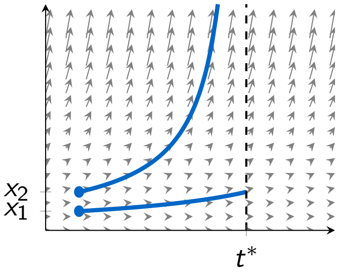
- Grönwall lemma
- exponential sensitivity



$$t \in \textit{ex-ivl}(x_1) \cap \textit{ex-ivl}(x_2) \implies |\varphi(x_1, t) - \varphi(x_2, t)| \in \mathcal{O}(\mathrm{e}^t)$$

Technical Lemmas

- Grönwall lemma
- exponential sensitivity
- same existence interval in neighborhood



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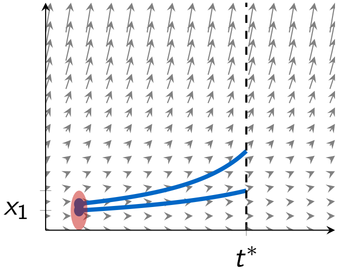

Technical Lemmas

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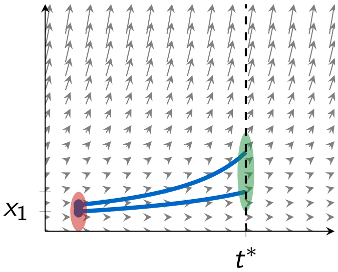
- same existence interval in neighborhood
- *continuous* φ *at* (x_1, t^*)



$$\forall \varepsilon > 0.\ \exists \delta.\ \varphi(U_\delta(x_1, t^*)) \subseteq U_\varepsilon(\varphi(x_1, t^*))$$

Technical Lemmas

- Grönwall lemma
- exponential sensitivity
- same existence interval in neighborhood
- *continuous* φ *at* (x_1, t^*)
- continuity w.r.t. right-hand side of ODE

$$\dot{x}(t) = f(x(t)); \qquad \dot{x}(t) = g(x(t))$$

$$|f - g| < \varepsilon \implies |\varphi_f(x_1, t) - \varphi_g(x_1, t)| \in \mathcal{O}(\mathrm{e}^t)$$

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $f'(x)$ derivative of $f : \mathbb{R} \to \mathbb{R}$

Variational Equation ($\mathbb{R}$)

$$\begin{cases} \dot{u}(t) = f'(\varphi(x_0, t)) \cdot u(t) \\ u(0) = 1 \end{cases}$$

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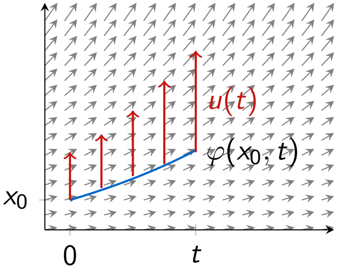


Differentiability

ODE $\dot{x}(t) =$ … ive of $f : \mathbb{R} \to \mathbb{R}$

Variational … $) \cdot u(t)$

17.6 Differentiability of the Flow

Now we return to the case of an autonomous differential equation $X' = F(X)$, where F is assumed to be C^1. Our aim is to show that the flow $\phi(t, X) = \phi_t(X)$ determined by this equation is a C^1 function of the two variables, and to identify $\partial\phi/\partial X$. We know, of course, that ϕ is continuously differentiable in the variable t, so it suffices to prove differentiability in X.

Toward that end let $X(t)$ be a particular solution of the system defined for t in a closed interval J about 0. Suppose $X(0) = X_0$. For each $t \in J$ let

$$A(t) = DF_{X(t)}.$$

That is, $A(t)$ denotes the Jacobian matrix of F at the point $X(t)$. Since F is C^1, $A(t)$ is continuous. We define the nonautonomous linear equation

$$U' = A(t)U.$$

This equation is known as the *variational equation* along the solution $X(t)$. From the previous section we know that the variational equation has a solution on all of J for every initial condition $U(0) = U_0$. Also, as in the autonomous case, solutions of this system satisfy the Linearity Principle.

The significance of this equation is that, if U_0 is small, then the function

$$t \to X(t) + U(t)$$

is a good approximation to the solution $X(t)$ of the original autonomous equation with initial value $X(0) = X_0 + U_0$.

To make this precise, suppose that $U(t, \xi)$ is the solution of the variational equation that satisfies $U(0, \xi) = \xi$ where $\xi \in \mathbb{R}^n$. If ξ and $X_0 + \xi$ belong to $\mathcal{O}$, let $Y(t, \xi)$ be the solution of the autonomous equation $X' = F(X)$ that satisfies $Y(0) = X_0 + \xi$.

Proposition. *Let J be the closed interval containing 0 on which $X(t)$ is defined. Then*

$$\lim_{\xi \to 0} \frac{|Y(t,\xi) - X(t) - U(t,\xi)|}{|\xi|}$$

converges to 0 uniformly for $t \in J$. □

This means that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|\xi| \le \delta$, then

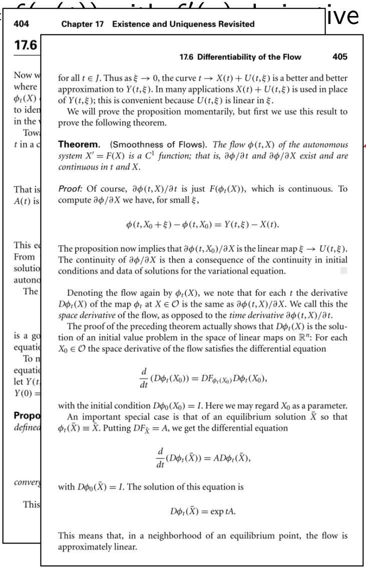
$$|Y(t,\xi) - (X(t) + U(t,\xi))| \le \epsilon|\xi|$$

Differentiability

ODE $\dot{x}(t) =$ … ive of $f : \mathbb{R} \to \mathbb{R}$

Variational

… $u(t)$



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17.6

17.6 Differentiability of the Flow 405

for all $t \in J$. Thus as $\xi \to 0$, the curve $t \to X(t) + U(t,\xi)$ is a better and better approximation to $Y(t,\xi)$. In many applications $X(t) + U(t,\xi)$ is used in place of $Y(t,\xi)$; this is convenient because $U(t,\xi)$ is linear in ξ.

We will prove the proposition momentarily, but first we use this result to prove the following theorem.

Theorem. (Smoothness of Flows). *The flow $\phi(t,X)$ of the autonomous system $X' = F(X)$ is a C^1 function; that is, $\partial\phi/\partial t$ and $\partial\phi/\partial X$ exist and are continuous in t and X.*

Proof: Of course, $\partial\phi(t,X)/\partial t$ is just $F(\phi_t(X))$, which is continuous. To compute $\partial\phi/\partial X$ we have, for small ξ,

$$\phi(t,X_0+\xi) - \phi(t,X_0) = Y(t,\xi) - X(t).$$

The proposition now implies that $\partial\phi(t,X_0)/\partial X$ is the linear map $\xi \to U(t,\xi)$. The continuity of $\partial\phi/\partial X$ is then a consequence of the continuity in initial conditions and data of solutions for the variational equation. ∎

Denoting the flow again by $\phi_t(X)$, we note that for each t the derivative $D\phi_t(X)$ of the map ϕ_t at $X \in \mathcal{O}$ is the same as $\partial\phi(t,X)/\partial X$. We call this the *space derivative* of the flow, as opposed to the *time derivative* $\partial\phi(t,X)/\partial t$.

The proof of the preceding theorem actually shows that $D\phi_t(X)$ is the solution of an initial value problem in the space of linear maps on $\mathbb{R}^n$: For each $X_0 \in \mathcal{O}$ the space derivative of the flow satisfies the differential equation

$$\frac{d}{dt}(D\phi_t(X_0)) = DF_{\phi_t(X_0)} D\phi_t(X_0),$$

with the initial condition $D\phi_0(X_0) = I$. Here we may regard X_0 as a parameter.

An important special case is that of an equilibrium solution $\bar{X}$ so that $\phi_t(\bar{X}) \equiv \bar{X}$. Putting $DF_{\bar{X}} = A$, we get the differential equation

$$\frac{d}{dt}(D\phi_t(\bar{X})) = AD\phi_t(\bar{X}),$$

with $D\phi_0(\bar{X}) = I$. The solution of this equation is

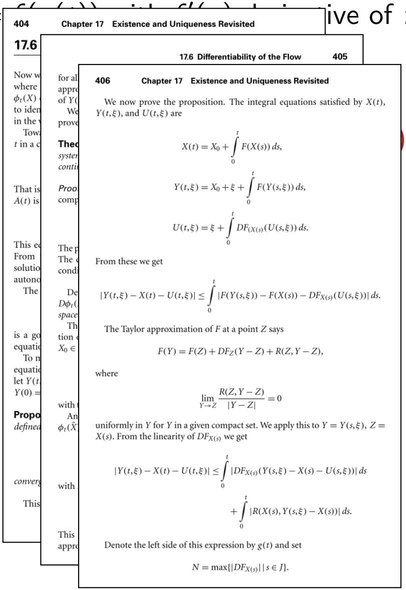
$$D\phi_t(\bar{X}) = \exp tA.$$

This means that, in a neighborhood of an equilibrium point, the flow is approximately linear.

Differentiability

ODE $\dot{x}(t) =$ … ive of $f : \mathbb{R} \to \mathbb{R}$

Variational …



404 Chapter 17 Existence and Uniqueness Revisited

17.6 Differentiability of the Flow 405

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We now prove the proposition. The integral equations satisfied by $X(t)$, $Y(t,\xi)$, and $U(t,\xi)$ are

$$X(t) = X_0 + \int_0^t F(X(s))\, ds,$$

$$Y(t,\xi) = X_0 + \xi + \int_0^t F(Y(s,\xi))\, ds,$$

$$U(t,\xi) = \xi + \int_0^t DF_{X(s)}(U(s,\xi))\, ds.$$

From these we get

$$|Y(t,\xi) - X(t) - U(t,\xi)| \leq \int_0^t |F(Y(s,\xi)) - F(X(s)) - DF_{X(s)}(U(s,\xi))|\, ds.$$

The Taylor approximation of F at a point Z says

$$F(Y) = F(Z) + DF_Z(Y - Z) + R(Z, Y - Z),$$

where

$$\lim_{Y \to Z} \frac{R(Z, Y - Z)}{|Y - Z|} = 0$$

uniformly in Y for Y in a given compact set. We apply this to $Y = Y(s,\xi)$, $Z = X(s)$. From the linearity of $DF_{X(s)}$ we get

$$|Y(t,\xi) - X(t) - U(t,\xi)| \leq \int_0^t |DF_{X(s)}(Y(s,\xi) - X(s) - U(s,\xi))|\, ds$$
$$+ \int_0^t |R(X(s), Y(s,\xi) - X(s))|\, ds.$$

Denote the left side of this expression by $g(t)$ and set

$$N = \max\{|DF_{X(s)}| \mid s \in J\}.$$

Differentiability

ODE $\dot{x}(t) =$ … ive of $f : \mathbb{R} \to \mathbb{R}$

Variational

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17.6

17.6 Differentiability of the Flow 405

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Then we have

$$g(t) \le N \int_0^t g(s)\,ds + \int_0^t |R(X(s), Y(s,\xi) - X(s))|\,ds.$$

Fix $\epsilon > 0$ and pick $\delta_0 > 0$ so small that

$$|R(X(s), Y(s,\xi) - X(s))| \le \epsilon |Y(s,\xi) - X(s)|$$

if $|Y(s,\xi) - X(s)| \le \delta_0$ and $s \in J$.

From Section 17.3 there are constants $K \ge 0$ and $\delta_1 > 0$ such that

$$|Y(s,\xi) - X(s)| \le |\xi| e^{Ks} \le \delta_0$$

if $|\xi| \le \delta_1$ and $s \in J$.

Assume now that $|\xi| \le \delta_1$. From the preceding, we find, for $t \in J$,

$$g(t) \le N \int_0^t g(s)\,ds + \int_0^t \epsilon |\xi| e^{Ks}\,ds,$$

so that

$$g(t) \le N \int_0^t g(s)\,ds + C\epsilon|\xi|$$

for some constant C depending only on K and the length of J. Applying Gronwall's Inequality, we obtain

$$g(t) \le C\epsilon e^{Nt} |\xi|$$

if $t \in J$ and $|\xi| \le \delta_1$. (Recall that δ_1 depends on ϵ.) Since ϵ is any positive number, this shows that $g(t)/|\xi| \to 0$ uniformly in $t \in J$, which proves the proposition.

EXERCISES

1. Write out the first few terms of the Picard iteration scheme for each of the following initial value problems. Where possible, use any method to find explicit solutions. Discuss the domain of the solution.

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $f'(x)$ derivative of $f : \mathbb{R} \to \mathbb{R}$

Variational Equation ($\mathbb{R}$)

$$\begin{cases} \dot{u}(t) = f'(\varphi(x_0, t)) \cdot u(t) \\ u(0) = 1 \end{cases}$$

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $f'(x)$ derivative of $f : \mathbb{R} \to \mathbb{R}$

Variational Equation ($\mathbb{R}$)

$$\begin{cases} \dot{u}(t) = f'(\varphi(x_0, t)) \cdot u(t) \\ u(0) = 1 \end{cases}$$

Theorem (derivative of flow)

$$\frac{\partial \varphi}{\partial x}(x_0, t) = u(t)$$

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $f'(x) : \mathbb{R}$

Variational Equation ($\mathbb{R}$)

$$\begin{cases} \dot{u}(t) = f'(\varphi(x_0, t)) \cdot u(t) \\ u(0) = 1 \end{cases}$$

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $\mathrm{D}f|_x : \mathbb{R}^{n \times n}$

Variational Equation ($\mathbb{R}^n$)

$$\begin{cases} \dot{u}(t) = \mathrm{D}f|_{\varphi(x_0,t)} \cdot u(t) \\ u(0) = 1_L \end{cases}$$

Differentiability

ODE $\dot{x}(t) = f(x(t))$ with $\mathrm{D}f|_x : \mathbb{R}^{n \times n}$

Variational Equation ($\mathbb{R}^n$)

$$\begin{cases} \dot{u}(t) = \mathrm{D}f|_{\varphi(x_0,t)} \cdot u(t) \\ u(0) = 1_L \end{cases}$$

requires: normed vector space of linear functions

- mathematics in Isabelle/HOL is type class based
  - topological, metric, vector, normed spaces are type classes

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requires: normed vector space of linear functions

- mathematics in Isabelle/HOL is type class based
  - topological, metric, vector, normed spaces are type classes
  - type of (bounded/continuous) linear functions

Structure

Flow

Dependence on Initial Condition

Numerics

Structure

Flow

Dependence on Initial Condition

Numerics

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- encode derivative of flow as linear ODE

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- [Immler, NFM2013/TACAS 2015]:
  verified numerical enclosures for solutions of ODEs

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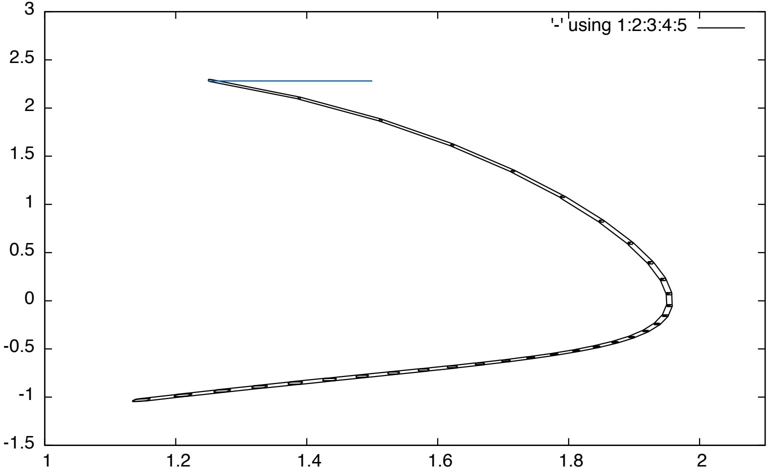
- [Immler, NFM2013/TACAS 2015]:
  verified numerical enclosures for solutions of ODEs
- van der Pol equations:
  $\dot{x} = y$
  $\dot{y} = (1 - x^2)y - x$
  $(x_0, y_0) = (1.25, 2.27)$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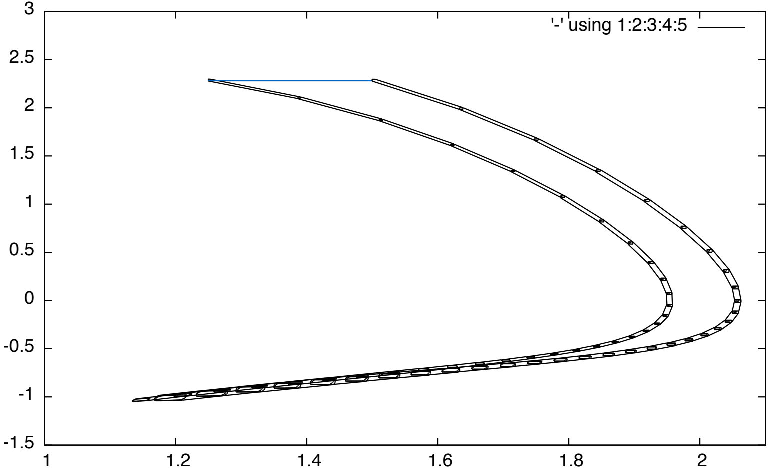
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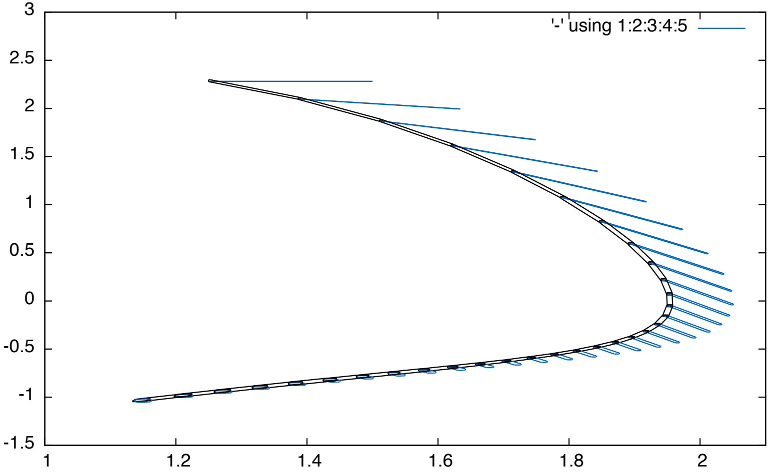

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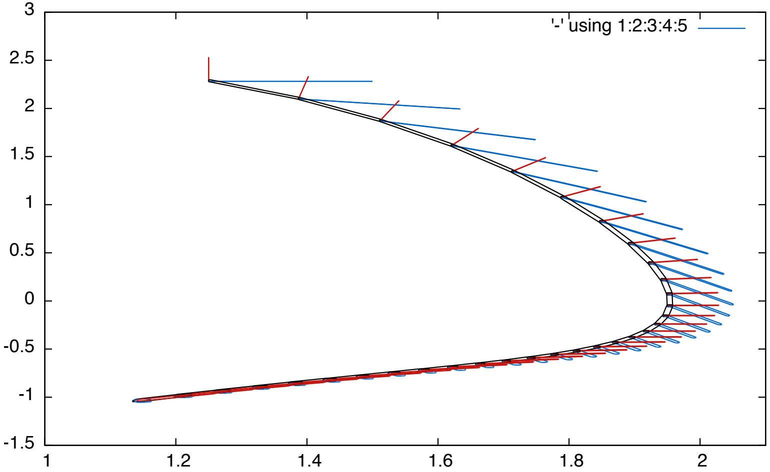
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Conclusion

- clean interface: flow φ, *ex-ivl*

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Conclusion

- clean interface: flow φ, *ex-ivl*
- hides tedious technical constructions
- employ existing verified algorithm for numerical enclosures
- general theory with concrete application:
  - Lorenz attractor
  - step towards formal verification of Tucker's proof