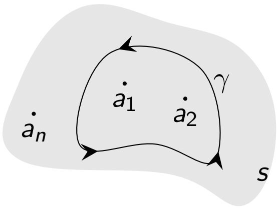
A Formal Proof of Cauchy's Residue Theorem

Wenda Li and Lawrence C. Paulson

University of Cambridge

{*wl302,lp15*}*@cam.ac.uk*

August 22, 2016



Informally, suppose f is holomorphic (i.e. complex differentiable) on an open set s except for a finite number of points $\{a_1, a_2, ..., a_n\}$ and γ is some closed path, Cauchy's residue theorem states

$$\oint_\gamma f = 2\pi i \sum_{k=1}^{n} n(\gamma, a_k)\mathrm{Res}(f, a_k)$$

where

- $n(\gamma, a_k)$ is the winding number of γ about a_k
- $\mathrm{Res}(f, a_k)$ is the residue of f at a_k

Overview

1. Background: the multivariate analysis library in Isabelle/HOL
2. Main proof of the residue theorem
3. Application to improper integrals
4. Corollaries: the argument principle and Rouché's theorem
5. Related work
6. Conclusion

Background: the multivariate analysis library

The multivariate analysis library in Isabelle/HOL

- about 70000 LOC (for now) on topology, analysis and linear algebra
- originates from John Harrison's work in HOL Light
  - $\mathbb{R}^n$ vs. type classes
  - scripted proofs vs. structured proofs

Background: contour integration

Contour integration is mathematically defined as

$$\oint_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

In Isabelle/HOL, we have

```
definition has_contour_integral ::
   "(complex ⇒ complex) ⇒ complex ⇒ (real ⇒ complex)
      ⇒ bool"
  where "(f has_contour_integral i) γ ≡
  ((λx. f(γ x) * vector_derivative γ (at x within {0..1}))
            has_integral i) {0..1}"
```

and also *contour_integral* of type

```
(real ⇒ complex) ⇒ (complex ⇒ complex) ⇒ complex
```

Background: valid paths

A valid path is a piecewise continuously differentiable function on [0..1].

```
lemma valid_path_def:
  fixes γ::"real ⇒ 'a::real_normed_vector"
  shows "valid_path γ ⟷ continuous_on {0..1} γ ∧
    (∃s. finite s ∧ γ C1_differentiable_on ({0..1} - s))"
```

Background: winding numbers

The winding number $n(\gamma, z)$ is the number of times the path γ travels counterclockwise around the point z:

$$n(\gamma, z) = \frac{1}{2\pi i} \oint_\gamma \frac{dw}{w - z}$$

A lemma to illustrate this definition is as follows:

```
lemma winding_number_valid_path:
  fixes γ::"real ⇒ complex" and z::complex
  assumes "valid_path γ" and "z ∉ path_image γ"
  shows "winding_number γ z
        = 1/(2*pi*i) * contour_integral γ (λw. 1/(w - z))"
```

Background: Cauchy's integral theorem

Cauchy's integral theorem states: given f is a holomorphic function on an open set s, which contains a closed path γ and its interior, then

$$\oint_\gamma f = 0$$

```
theorem Cauchy_theorem_global:
  fixes s::"complex set" and f::"complex ⇒ complex"
    and γ::"real ⇒ complex"
  assumes "open s" and "f holomorphic_on s"
    and "valid_path γ" and "pathfinish γ = pathstart γ"
    and "path_image γ ⊆ s"
    and "⋀w. w ∉ s ⟹ winding_number γ w = 0"
  shows "(f has_contour_integral 0) γ"
```

Main proof: singularities

Cauchy's integral theorem does not apply when there are singularities.

For example, consider $f(w) = \frac{1}{w}$ so that f has a pole at $w = 0$, and γ is the circular path $\gamma(t) = e^{2\pi it}$:

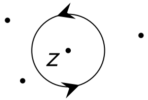
$$\oint_\gamma \frac{dw}{w} = \int_0^1 \frac{1}{e^{2\pi it}} \left(\frac{d}{dt} e^{2\pi it} \right) dt = \int_0^1 2\pi i dt = 2\pi i \neq 0$$

Main proof: residue

The residue of f at z can be defined as

$$\mathrm{Res}(f, z) = \frac{1}{2\pi i} \oint_{\mathrm{circ}(z)} f$$

where $\mathrm{circ}(z)$ is a small counterclockwise circular path around z.



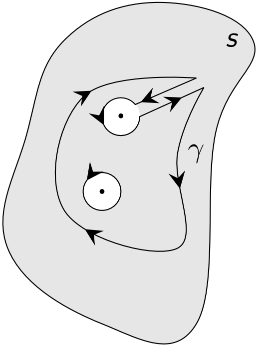

```
definition residue
    ::"(complex ⇒ complex) ⇒ complex ⇒ complex"
where "residue f z = (SOME int. ∃δ>0. ∀ε>0. ε<δ ⟶
    (f has_contour_integral 2*pi*i*int) (circlepath z ε))"
```

Main proof: Cauchy's residue theorem

```
lemma Residue_theorem:
  fixes s pts::"complex set" and f::"complex ⇒ complex"
    and γ::"real ⇒ complex"
  assumes "open s" and "connected s"
      and "finite pts" and "f holomorphic_on s-pts"
      and "valid_path γ" and "pathfinish γ = pathstart γ"
      and "path_image γ ⊆ s-pts"
      and "∀z. (z ∉ s) ⟶ winding_number γ z = 0"
  shows "contour_integral γ f
    = 2*pi*i*(∑p∈pts. winding_number γ p * residue f p)"
```

Main proof: remarks

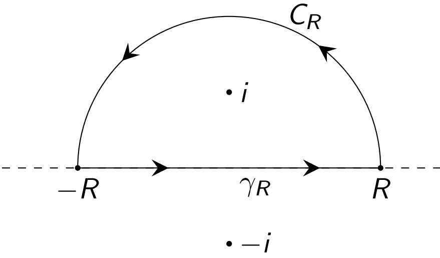
- the core idea to induct on the number of singularity points
- gap in informal proofs



Application to improper integrals

The residue theorem can be used to solve improper integrals:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$$



Proof: let

$$f(z) = \frac{1}{z^2+1.}$$

since $\lim_{R\to\infty} \oint_{C_R} f = 0$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{R\to\infty} \oint_{\gamma_R + C_R} f$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$$

$$\Leftrightarrow \lim_{R\to\infty} \oint_{\gamma_R} f = \pi$$

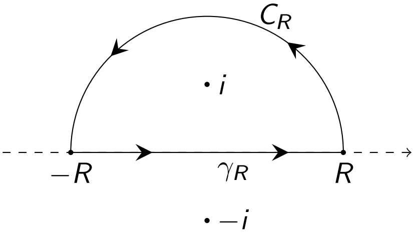
$$\Leftrightarrow \lim_{R\to\infty} (\oint_{\gamma_R} f + \oint_{C_R} f) = \pi$$

$$\Leftrightarrow \lim_{R\to\infty} \oint_{\gamma_R + C_R} f$$

$$= 2\pi i \mathrm{Res}(\gamma_R + C_R, i) = \pi$$

```
def f≡"λx::real. 1/(x^2+1)"
def f'≡"λx::complex. 1/(x^2+1)"

have "((λR. integral {- R..R} f) ⟶ pi) at_top
    = ((λR. contour_integral (γR R) f')
      ⟶ pi) at_top"

also have "... = ((λR. contour_integral (CR R) f'
    + contour_integral (γR R) f') ⟶ pi) at_top"

also have "... = ((λR. contour_integral
              (CR R +++ γR R) f') ⟶ pi) at_top"

also have "..."
proof -
  have "contour_integral (CR R +++ γR R) f' = pi"
    when "R>1" for R
  then show ?thesis
qed

finally have "((λR. integral {- R..R}
              (λx. 1 / (x^2 + 1))) ⟶ pi) at_top"
```

Application to improper integrals (3)



lemma *improper_Ex:*

```
"Lim at_top (λR. integral {- R..R} (λx. 1/(x²+1))) = pi"
```

Remarks

- about 300 LOC
- the main difficulty is to show $n(\gamma_R + C_R, i) = 1$ and $n(\gamma_R + C_R, -i) = 0$

Corollaries: the argument principle

The argument principle states: suppose f is holomorphic on a connected open set s except for a finite number of poles and γ is a valid closed path, then

$$\oint_{\gamma} \frac{f'}{f} = 2\pi i \left(\sum_{z \in zeros} n(\gamma, z) zorder(f, z) - \sum_{z \in poles} n(\gamma, z) porder(f, z) \right)$$

where

- f' is the first derivative of f;
- $zorder(f, z)$ and $porder(f, z)$ are the order of a zero and a pole respectively;
- $zeros$ and $poles$ are the zeros and poles respectively of f.

Corollaries: Rouché's theorem

Given two functions f and g holomorphic on a connected open set containing a valid path γ, if

$$|f(w)| > |g(w)| \qquad \forall w \in \text{image of } \gamma$$

then Rouché's Theorem states

$$\sum_{z \in zeros(f)} n(\gamma, z) zorder(f, z) = \sum_{zeros(f+g)} n(\gamma, z) zorder(f+g, z)$$

Big picture

Cauchy's residue theorem

↓

Rouché's theorem

↓

Roots of a polynomial depend continuously on its coefficients

↓

Decision procedure to solve first-order formulas over real numbers

in progress

Related work

To the best of our knowledge, our formalization of Cauchy's residue theorem is novel among major proof assistants.

- HOL Light: comprehensive library for complex analysis, to which it should be not hard to port our result;
- Coq: Coquelicot and C-Corn, but they are mainly about real analysis and some fundamental theorems (e.g. Cauchy's integral theorem) are not yet available.

Conclusion

To conclude, I have mainly covered

- the multivariate analysis library in Isabelle/HOL
- the residue theorem
- application to improper integrals
- corollaries: the argument principle and Rouché's theorem

Thank you for your attention!