Equational Reasoning with Applicative Functors

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Equational Reasoning with Applicative Functors

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model effects

- state
- probabilities
- error
- non-determinism
- 1 2 3 4 ··· streams

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$f\ (g\ \bullet) = h\ (k\ \bullet)$

$$\begin{tikzcd} \bullet \arrow[r, "k"] \arrow[d, "g"'] & \blacksquare \arrow[d, "h"] \\ \blacktriangle \arrow[r, "f"] & \bigstar \end{tikzcd}$$

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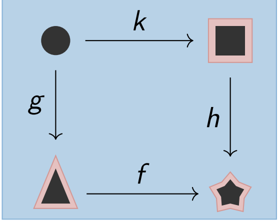
$\text{pure } f \diamond (g\ \bullet) = \text{pure } h \diamond (k\ \bullet)$



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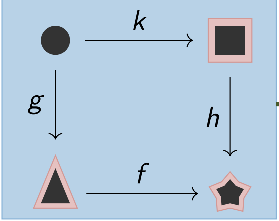
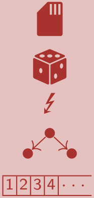
$\mathsf{pure}\ f \diamond (g\ \bullet) = \mathsf{pure}\ h \diamond (k\ \bullet)$



don't care

model effects



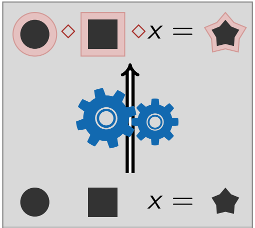
state
probabilities
error
non-determinism
streams

Contributions

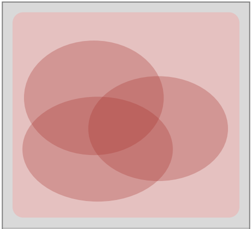
- Isabelle/HOL package for reasoning about applicative effects

```
applicative state
for
  pure: pure_state
  ap: ap_state
proof (prove)
goal (4 subgoals):
 1. ⋀f x. pure f ◇ pure x = pure (
 2. ⋀g f x. pure (λg f x. g (f x))
 3. ⋀x. pure (λx. x) ◇ x = x
 4. ⋀f x. f ◇ pure x = pure (λf. f
```

functor registration



proof tactic

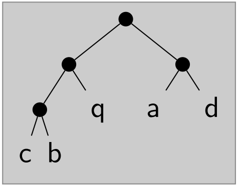


classify effects

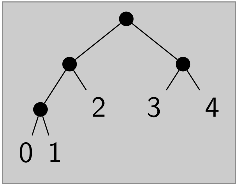
- Meta theory formalised and algorithms verified

- Used in several examples and case studies

Task: Label a binary tree with distinct numbers!



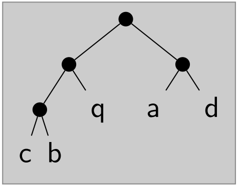
$\xrightarrow{\quad\text{lbl}\quad}$



datatype α *tree* $=$
  L α | N (α *tree*) (α *tree*)

Example suggested by G. Hutton, D. Fulger: Reasoning about effects: seeing the wood through the trees. TFP 2008

Task: Label a binary tree with distinct numbers!



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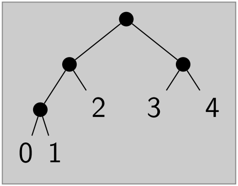
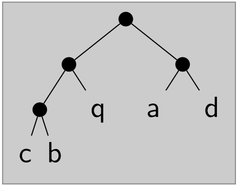


datatype α *tree* =
  L α | N (α *tree*) (α *tree*)

lbl :: α *tree* ⇒ *nat tree*

Example suggested by G. Hutton, D. Fulger: Reasoning about effects: seeing the wood through the trees. TFP 2008

Task: Label a binary tree with distinct numbers!



$$\xrightarrow{\quad \text{lbl} \quad}$$

datatype α *tree* $=$
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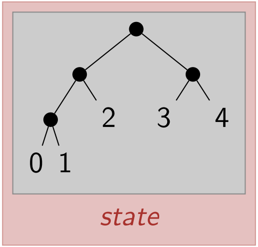


lbl :: α *tree* $\Rightarrow$ *nat tree state* where α *state* $=$ *nat* $\Rightarrow \alpha \times$ *nat*

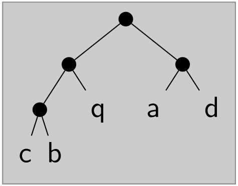
monadic α $M = \alpha$ *state*

return :: $\alpha \Rightarrow \alpha$ M
($\gg=$) :: α $M \Rightarrow (\alpha \Rightarrow \beta$ $M) \Rightarrow \beta$ M

lbl (L _) = fresh $\gg=$ $\lambda x'$. return (L x')
lbl (N l r) =
lbl l $\gg=$ $\lambda l'$. lbl r $\gg=$ $\lambda r'$. return (N l' r')

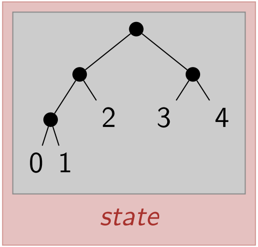
Example suggested by G. Hutton, D. Fulger: Reasoning about effects: seeing the wood through the trees. TFP 2008

Task: Label a binary tree with distinct numbers!



$\xrightarrow{\mathsf{lbl}}$

datatype α *tree* =
  L α | N (α *tree*) (α *tree*)



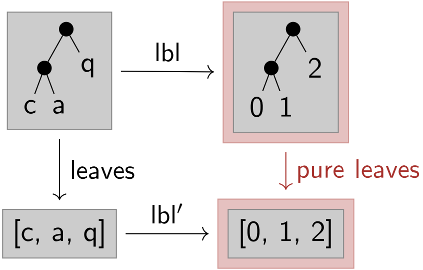
lbl :: α *tree* $\Rightarrow$ *nat tree state*   where α *state* = *nat* $\Rightarrow \alpha \times$ *nat*

monadic $\quad \alpha\ M = \alpha$ *state*
return :: $\alpha \Rightarrow \alpha\ M$
$(\ggg\!=)$ :: $\alpha\ M \Rightarrow (\alpha \Rightarrow \beta\ M) \Rightarrow \beta\ M$
lbl (L _) = fresh $\ggg\!= \lambda x'$. return (L x')
lbl (N l r) =
lbl l $\ggg\!= \lambda l'$. lbl r $\ggg\!= \lambda r'$. return (N l' r')

applicative $\quad \alpha\ F = \alpha$ *state*
pure :: $\alpha \Rightarrow \alpha\ F$
$(\diamond)$ :: $(\alpha \Rightarrow \beta)\ F \Rightarrow \alpha\ F \Rightarrow \beta\ F$
lbl (L _) = pure L $\diamond$ fresh
lbl (N l r) = pure N $\diamond$ lbl l $\diamond$ lbl r

Example suggested by G. Hutton, D. Fulger: Reasoning about effects: seeing the wood through the trees. TFP 2008

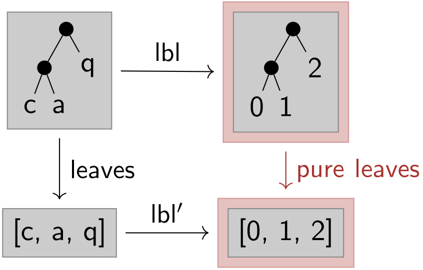
Labelling trees and lists



$$\begin{aligned}
&\mathsf{leaves} :: \alpha\ \mathit{tree} \Rightarrow \alpha\ \mathit{list}\\
&\mathsf{leaves}\ (\mathsf{L}\ x) \quad = x \cdot [\,]\\
&\mathsf{leaves}\ (\mathsf{N}\ l\ r) = \mathsf{leaves}\ l \mathbin{+\!\!+} \mathsf{leaves}\ r
\end{aligned}$$

$$\begin{aligned}
&\mathsf{lbl}' :: \alpha\ \mathit{list} \Rightarrow \mathit{nat}\ \mathit{list}\ \mathit{state}\\
&\mathsf{lbl}'\ [\,] \qquad\quad = \mathsf{pure}\ [\,]\\
&\mathsf{lbl}'\ (_ \cdot xs) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{lbl}'\ xs
\end{aligned}$$

Labelling trees and lists



$$
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\end{aligned}
$$

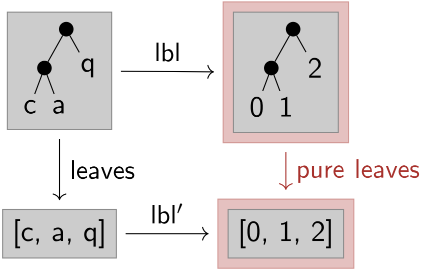
$$
\begin{aligned}
&\text{lbl}' :: \alpha\ \textit{list} \Rightarrow \textit{nat list state}\\
&\text{lbl}'\ [\,] \qquad\quad = \text{pure}\ [\,]\\
&\text{lbl}'\ (\_ \cdot xs) = \text{pure}\ (\cdot) \diamond \text{fresh} \diamond \text{lbl}'\ xs
\end{aligned}
$$

Lemma: $\text{pure leaves} \diamond \text{lbl}\ t = \text{lbl}'\ (\text{leaves}\ t)$

Proof by induction on t.

Case $\mathsf{L}\ x$: $\text{pure leaves} \diamond \text{lbl}\ (\mathsf{L}\ x) \qquad = \text{lbl}'\ (\text{leaves}\ (\mathsf{L}\ x))$

Labelling trees and lists



$\mathsf{leaves} :: \alpha\ \mathit{tree} \Rightarrow \alpha\ \mathit{list}$
$\mathsf{leaves}\ (\mathsf{L}\ x) \quad = x \cdot [\,]$
$\mathsf{leaves}\ (\mathsf{N}\ l\ r) = \mathsf{leaves}\ l +\!\!+\ \mathsf{leaves}\ r$

$\mathsf{lbl}' :: \alpha\ \mathit{list} \Rightarrow \mathit{nat}\ \mathit{list}\ \mathit{state}$
$\mathsf{lbl}'\ [\,] \qquad = \mathsf{pure}\ [\,]$
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Lemma: $\mathsf{pure}\ \mathsf{leaves} \diamond \mathsf{lbl}\ t = \mathsf{lbl}'\ (\mathsf{leaves}\ t)$

Proof by induction on t.

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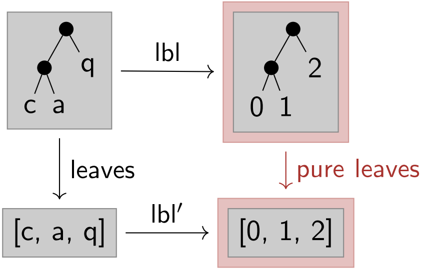
$\mathsf{pure}\ \mathsf{leaves} \diamond (\mathsf{pure}\ \mathsf{L} \diamond \mathsf{fresh}) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$

$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) = (\cdot)\ x\ [\,]$

Labelling trees and lists



leaves :: α *tree* $\Rightarrow$ α *list*
leaves (L x) $= x \cdot [\,]$
leaves (N l r) $=$ leaves l $+\!+$ leaves r

lbl′ :: α *list* $\Rightarrow$ *nat list state*
lbl′ [ ] $=$ pure [ ]
lbl′ ($\_ \cdot xs$) $=$ pure ($\cdot$) $\diamond$ fresh $\diamond$ lbl′ xs

Lemma: pure leaves $\diamond$ lbl t $=$ lbl′ (leaves t)

Proof by induction on t.

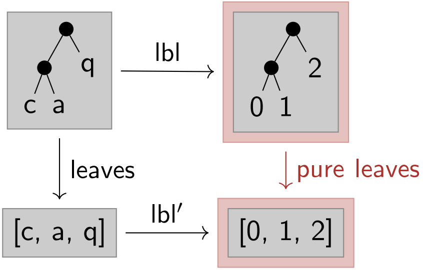
Case L x: pure leaves $\diamond$ lbl (L x) $=$ lbl′ (leaves (L x))

pure leaves $\diamond$ (pure L $\diamond$ fresh) $=$ pure ($\cdot$) $\diamond$ fresh $\diamond$ pure [ ]

holds by the applicative laws $\Uparrow$

$\forall x.$ leaves (L x) $=$ ($\cdot$) x [ ]

Labelling trees and lists



$\mathsf{leaves} :: \alpha\ \mathit{tree} \Rightarrow \alpha\ \mathit{list}$

$\mathsf{leaves}\ (\mathsf{L}\ x) \quad = x \cdot [\,]$

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$\mathsf{lbl}' :: \alpha\ \mathit{list} \Rightarrow \mathit{nat}\ \mathit{list}\ \mathit{state}$

$\mathsf{lbl}'\ [\,] \qquad = \mathsf{pure}\ [\,]$

$\mathsf{lbl}'\ (\_ \cdot xs) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{lbl}'\ xs$

Lemma: $\mathsf{pure}\ \mathsf{leaves} \diamond \mathsf{lbl}\ t = \mathsf{lbl}'\ (\mathsf{leaves}\ t)$

Proof by induction on t.

Case L x: $\mathsf{pure}\ \mathsf{leaves} \diamond \mathsf{lbl}\ (\mathsf{L}\ x) = \mathsf{lbl}'\ (\mathsf{leaves}\ (\mathsf{L}\ x))$

$\mathsf{pure}\ \mathsf{leaves} \diamond (\mathsf{pure}\ \mathsf{L} \diamond \mathsf{fresh}) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$

holds by the applicative laws $\Uparrow$ `apply applicative_lifting`

$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) = (\cdot)\ x\ [\,]$

Lifting equations over applicative functors

[Hinze 2010]

$$\text{pure leaves} \diamond (\text{pure L} \diamond \text{fresh}) \;=\; \text{pure} \; (\cdot) \diamond \text{fresh} \diamond \text{pure} \; [\,]$$

$$\Uparrow$$

$$\forall x.\ \text{leaves} \; (\text{L} \; x) \;=\; x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

$$\mathsf{pure}\ \mathsf{leaves} \diamond (\mathsf{pure}\ \mathsf{L} \diamond \mathsf{fresh}) \;=\; \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$$

$$\Uparrow$$

ML

syntactic formalisation

follows

$$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) \;=\; x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

$$\text{pure leaves} \diamond (\text{pure L} \diamond \text{fresh}) = \text{pure } (\cdot) \diamond \text{fresh} \diamond \text{pure } []$$

$\|$ 1. Convert to canonical form $\|$

$$\text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond \text{fresh} = \text{pure } (\lambda x.\ x \cdot []) \diamond \text{fresh}$$

$\Uparrow$

$$\forall x.\ \text{leaves } (\text{L } x) = x \cdot []$$

Lifting equations over applicative functors

[Hinze 2010]

Canonical form [McBride, Paterson]

applicative expression $\mapsto$ $\mathsf{pure}\ f \diamond x_1 \diamond x_2 \diamond \ldots \diamond x_n$

$$\mathsf{pure\ leaves} \diamond (\mathsf{pure\ L} \diamond \mathsf{fresh}) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ []$$

1. Convert to canonical form

$$\mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x)) \diamond \mathsf{fresh} = \mathsf{pure}\ (\lambda x.\ x \cdot []) \diamond \mathsf{fresh}$$

$$\Uparrow$$

$$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) = x \cdot []$$

Lifting equations over applicative functors

[Hinze 2010]

pure function   opaque arguments

Canonical form [McBride, Paterson]

applicative expression $\mapsto$ $\boxed{\mathsf{pure}\ f} \diamond \boxed{x_1} \diamond \boxed{x_2} \diamond \boxed{\ldots} \diamond \boxed{x_n}$

$$\mathsf{pure\ leaves} \diamond (\mathsf{pure\ L} \diamond \mathsf{fresh}) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$$

1. Convert to canonical form

$$\boxed{\mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x))} \diamond \boxed{\mathsf{fresh}} = \boxed{\mathsf{pure}\ (\lambda x.\ x \cdot [\,])} \diamond \boxed{\mathsf{fresh}}$$

$$\Uparrow$$

$$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) = x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

Canonical form [McBride, Paterson]

pure function — opaque arguments

applicative expression $\mapsto$ $\boxed{\mathsf{pure}\ f} \diamond \boxed{x_1} \diamond \boxed{x_2} \diamond \boxed{\ldots} \diamond \boxed{x_n}$

$$\mathsf{pure\ leaves} \diamond (\mathsf{pure\ L} \diamond \mathsf{fresh}) \;=\; \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$$

$\|$ 1. Convert to canonical form $\|$

$$\boxed{\mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x))} \diamond \boxed{\mathsf{fresh}} \;=\; \boxed{\mathsf{pure}\ (\lambda x.\ x \cdot [\,])} \diamond \boxed{\mathsf{fresh}}$$

2. Generalise opaque arguments $\Uparrow$

$$\boxed{\forall X.}\ \mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x)) \diamond \boxed{X} \;=\; \mathsf{pure}\ (\lambda x.\ x \cdot [\,]) \diamond \boxed{X}$$

$\Uparrow$

$$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) \;=\; x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

pure function

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Canonical form [McBride, Paterson]

applicative expression $\mapsto$ $\boxed{\mathsf{pure}\ f} \diamond \boxed{x_1} \diamond \boxed{x_2} \diamond \boxed{\ldots} \diamond \boxed{x_n}$

$$\mathsf{pure\ leaves} \diamond (\mathsf{pure\ L} \diamond \mathsf{fresh}) = \mathsf{pure}\ (\cdot) \diamond \mathsf{fresh} \diamond \mathsf{pure}\ [\,]$$

$\Vert$ 1. Convert to canonical form $\Vert$

$$\boxed{\mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x))} \diamond \boxed{\mathsf{fresh}} = \boxed{\mathsf{pure}\ (\lambda x.\ x \cdot [\,])} \diamond \boxed{\mathsf{fresh}}$$

2. Generalise opaque arguments $\Uparrow$

$$\boxed{\forall X.}\ \mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x)) \diamond \boxed{X} = \mathsf{pure}\ (\lambda x.\ x \cdot [\,]) \diamond \boxed{X}$$

3. Equality is a congruence $\Uparrow$

$$\forall X.\ \mathsf{pure}\ (\lambda x.\ \mathsf{leaves}\ (\mathsf{L}\ x)) \diamond X = \mathsf{pure}\ (\lambda x.\ x \cdot [\,]) \diamond X$$

$\Uparrow$

$$\forall x.\ \mathsf{leaves}\ (\mathsf{L}\ x) = x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

Canonical form [McBride, Paterson]

pure function / opaque arguments

applicative expression $\mapsto$ $\text{pure } f \diamond x_1 \diamond x_2 \diamond \ldots \diamond x_n$

$$\text{pure leaves} \diamond (\text{pure L} \diamond \text{fresh}) = \text{pure } (\cdot) \diamond \text{fresh} \diamond \text{pure } [\,]$$

1. Convert to canonical form

$$\text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond \text{fresh} = \text{pure } (\lambda x.\ x \cdot [\,]) \diamond \text{fresh}$$

2. Generalise opaque arguments

$$\forall X.\ \text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond X = \text{pure } (\lambda x.\ x \cdot [\,]) \diamond X$$

3. Equality is a congruence

$$\forall X.\ \text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond X = \text{pure } (\lambda x.\ x \cdot [\,]) \diamond X$$

4. Use extensionality

$$\forall x.\ \text{leaves } (\text{L } x) = x \cdot [\,]$$

Lifting equations over applicative functors

[Hinze 2010]

pure function

opaque arguments

Canonical form [McBride, Paterson]

applicative expression $\mapsto$ $\boxed{\text{pure } f} \diamond \boxed{x_1} \diamond \boxed{x_2} \diamond \boxed{\ldots} \diamond \boxed{x_n}$

$$\text{pure leaves} \diamond (\text{pure L} \diamond \text{fresh}) = \text{pure } (\cdot) \diamond \text{fresh} \diamond \text{pure } [\,]$$

1. Convert to canonical form

$$\boxed{\text{pure } (\lambda x.\ \text{leaves } (\text{L } x))} \diamond \boxed{\text{fresh}} = \boxed{\text{pure } (\lambda x.\ x \cdot [\,])} \diamond \boxed{\text{fresh}}$$

2. Generalise opaque arguments

$$\boxed{\forall X.}\ \text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond \boxed{X} = \text{pure } (\lambda x.\ x \cdot [\,]) \diamond \boxed{X}$$

3. Equality is a congruence

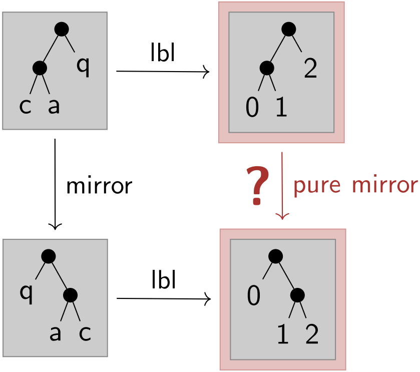
Same opaque args. on both sides!

$$\forall X.\ \text{pure } (\lambda x.\ \text{leaves } (\text{L } x)) \diamond X = \text{pure } (\lambda x.\ x \cdot [\,]) \diamond X$$

4. Use extensionality

$$\forall x.\ \text{leaves } (\text{L } x) = x \cdot [\,]$$

Tree mirroring



$$
\begin{aligned}
&\text{mirror} :: \alpha\ \textit{tree} \Rightarrow \alpha\ \textit{tree}\\
&\text{mirror}\ (\mathsf{L}\ x) \quad\ = \mathsf{L}\ x\\
&\text{mirror}\ (\mathsf{N}\ l\ r) = \mathsf{N}\ (\text{mirror}\ r)\ (\text{mirror}\ l)
\end{aligned}
$$

$$
\begin{aligned}
&\text{lbl} :: \alpha\ \textit{tree} \Rightarrow \textit{nat tree state}\\
&\text{lbl}\ (\mathsf{L}\ \_) \quad\ = \text{pure}\ \mathsf{L} \diamond \text{fresh}\\
&\text{lbl}\ (\mathsf{N}\ l\ r) = \text{pure}\ \mathsf{N} \diamond \text{lbl}\ l \diamond \text{lbl}\ r
\end{aligned}
$$

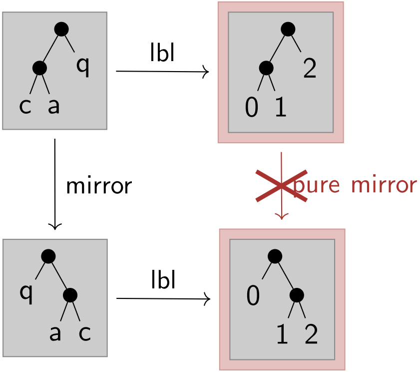
Lemma: $\text{lbl}\ (\text{mirror}\ t) = \text{pure mirror} \diamond \text{lbl}\ t$

Proof by induction on t.

Case $\mathsf{N}\ l\ r$:

$$
\begin{aligned}
&\boxed{\text{pure}\ (\lambda r'\ l'.\ \mathsf{N}\ (\text{mirror}\ r')\ (\text{mirror}\ l'))} \diamond \boxed{\text{lbl}\ r} \diamond \boxed{\text{lbl}\ l}\\
\overset{?}{=}\ &\boxed{\text{pure}\ (\lambda l'\ r'.\ \text{mirror}\ (\mathsf{N}\ l'\ r'))} \diamond \boxed{\text{lbl}\ l} \diamond \boxed{\text{lbl}\ r}
\end{aligned}
$$

Tree mirroring



$$
\begin{aligned}
&\text{mirror} :: \alpha\ \textit{tree} \Rightarrow \alpha\ \textit{tree}\\
&\text{mirror}\ (\text{L}\ x) \quad = \text{L}\ x\\
&\text{mirror}\ (\text{N}\ l\ r) = \text{N}\ (\text{mirror}\ r)\ (\text{mirror}\ l)
\end{aligned}
$$

$$
\begin{aligned}
&\text{lbl} :: \alpha\ \textit{tree} \Rightarrow \textit{nat tree state}\\
&\text{lbl}\ (\text{L}\ \_) \quad = \text{pure}\ \text{L} \diamond \text{fresh}\\
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\end{aligned}
$$

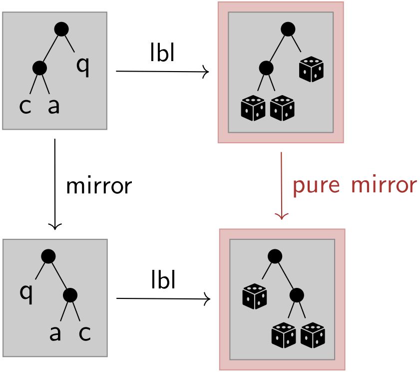
Lemma: $\text{lbl}\ (\text{mirror}\ t) = \text{pure mirror} \diamond \text{lbl}\ t$

Proof by induction on t.

Case N l r:

$$
\begin{aligned}
&\boxed{\text{pure}\ (\lambda r'\ l'.\ \text{N}\ (\text{mirror}\ r')\ (\text{mirror}\ l'))} \diamond \boxed{\text{lbl}\ r} \diamond \boxed{\text{lbl}\ l}\\
\stackrel{?}{=}\ &\boxed{\text{pure}\ (\lambda l'\ r'.\ \text{mirror}\ (\text{N}\ l'\ r'))} \diamond \boxed{\text{lbl}\ l} \diamond \boxed{\text{lbl}\ r}
\end{aligned}
$$

Tree mirroring and random labels



$\text{mirror} :: \alpha\ \mathit{tree} \Rightarrow \alpha\ \mathit{tree}$
$\text{mirror}\ (\text{L}\ x) \quad = \text{L}\ x$
$\text{mirror}\ (\text{N}\ l\ r) = \text{N}\ (\text{mirror}\ r)\ (\text{mirror}\ l)$

$\text{lbl} :: \alpha\ \mathit{tree} \Rightarrow \mathit{nat}\ \mathit{tree}\ \mathit{probability}$
$\text{lbl}\ (\text{L}\ \_) \quad = \text{pure}\ \text{L} \diamond \text{fresh}$
$\text{lbl}\ (\text{N}\ l\ r) = \text{pure}\ \text{N} \diamond \text{lbl}\ l \diamond \text{lbl}\ r$

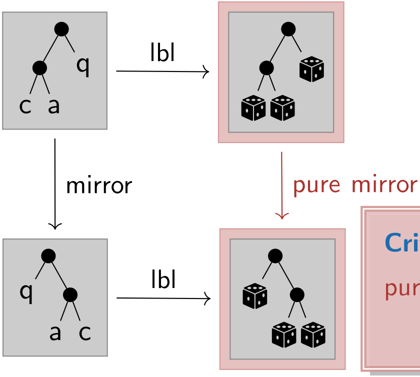
Lemma: $\text{lbl}\ (\text{mirror}\ t) = \text{pure}\ \text{mirror} \diamond \text{lbl}\ t$ **if effects commute**

Proof by induction on t.

Case $\text{N}\ l\ r$:

$$\boxed{\text{pure}\ (\lambda r'\ l'.\ \text{N}\ (\text{mirror}\ r')\ (\text{mirror}\ l'))} \diamond \boxed{\text{lbl}\ r} \diamond \boxed{\text{lbl}\ l}$$
$$= \boxed{\text{pure}\ (\lambda l'\ r'.\ \text{mirror}\ (\text{N}\ l'\ r'))} \diamond \boxed{\text{lbl}\ l} \diamond \boxed{\text{lbl}\ r}$$

Tree mirroring and random labels



$$
\begin{aligned}
&\mathsf{mirror} :: \alpha\ \mathit{tree} \Rightarrow \alpha\ \mathit{tree}\\
&\mathsf{mirror}\ (\mathsf{L}\ x) \quad = \mathsf{L}\ x\\
&\mathsf{mirror}\ (\mathsf{N}\ l\ r) = \mathsf{N}\ (\mathsf{mirror}\ r)\ (\mathsf{mirror}\ l)
\end{aligned}
$$

Criterion for commutative effects:

$$\mathsf{pure}\ (\lambda f\ x\ y.\ f\ y\ x) \diamond f \diamond x \diamond y \ = \ f \diamond y \diamond x$$

$$\mathbf{C} \qquad f\ \ x\ \ y \ = \ f\ \ y\ \ x$$

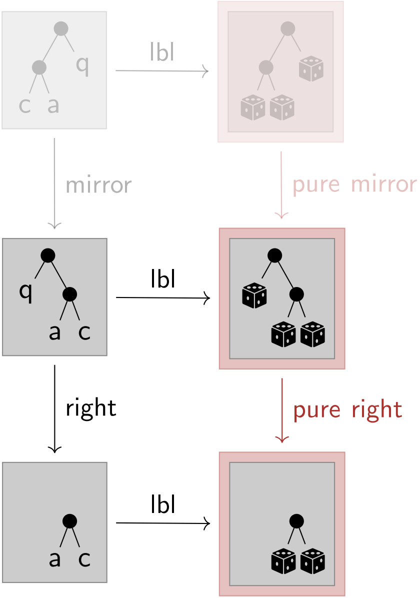
Lemma: $\mathsf{lbl}\ (\mathsf{mirror}\ t) = \mathsf{pure}\ \mathsf{mirror} \diamond \mathsf{lbl}\ t$ **if effects commute**

Proof by induction on t.

Case $\mathsf{N}\ l\ r$:

$$
\begin{aligned}
& \mathsf{pure}\ (\lambda r'\ l'.\ \mathsf{N}\ (\mathsf{mirror}\ r')\ (\mathsf{mirror}\ l')) \diamond \mathsf{lbl}\ r \diamond \mathsf{lbl}\ l\\
= \ & \mathsf{pure}\ (\lambda l'\ r'.\ \mathsf{mirror}\ (\mathsf{N}\ l'\ r')) \diamond \mathsf{lbl}\ l \diamond \mathsf{lbl}\ r
\end{aligned}
$$

Subtrees



Lemma:

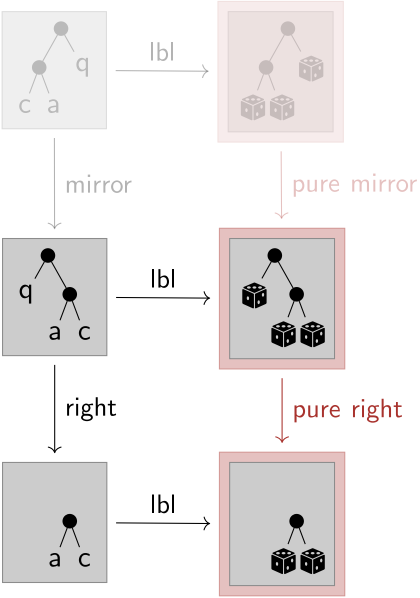
$\mathsf{lbl}\ (\mathsf{right}\ t) = \mathsf{pure}\ \mathsf{right} \diamond \mathsf{lbl}\ t$

Proof by case analysis on t.

Case N l r:

$$\mathsf{pure}\ (\lambda r'.\ \ r') \diamond \boxed{\mathsf{lbl}\ r}$$
$$\stackrel{?}{=}$$
$$\mathsf{pure}\ (\lambda\_\ r'.\ r') \diamond \boxed{\mathsf{lbl}\ l} \diamond \boxed{\mathsf{lbl}\ r}$$

Subtrees



Criterion for omissible effects:

$$\text{pure } (\lambda x\ y.\ x) \diamond x \diamond y = x$$
$$\mathbf{K}\quad x\quad y = x$$

Lemma: **if effects are omissible**

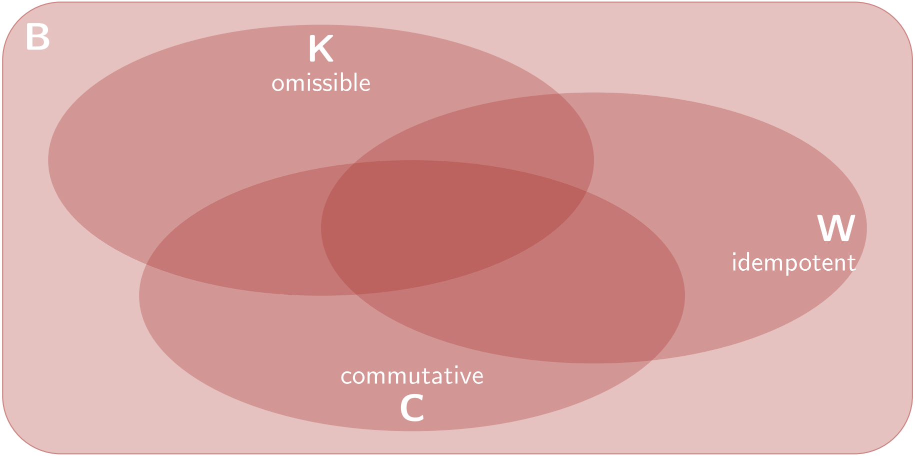
$$\text{lbl } (\text{right } t) = \text{pure right} \diamond \text{lbl } t$$

Proof by case analysis on t.

Case N l r:

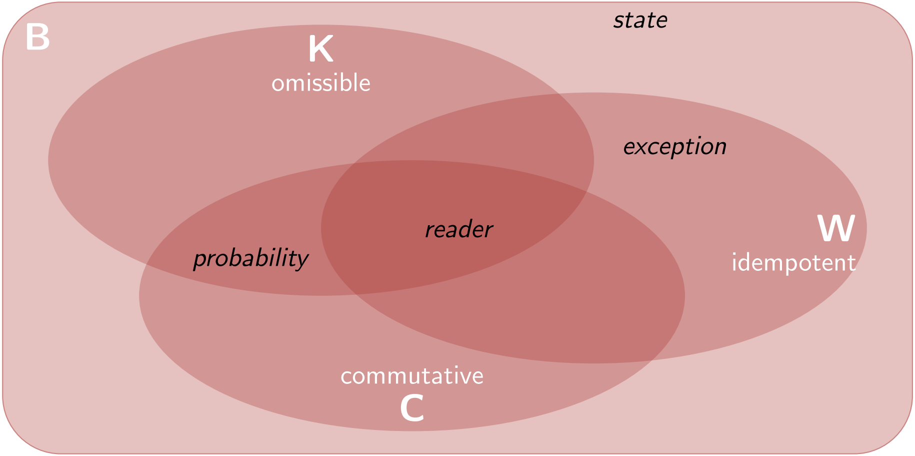
$$\text{pure } (\lambda r'.\ \ r') \qquad \diamond \boxed{\text{lbl } r}$$
$$=$$
$$\text{pure } (\lambda\_\ r'.\ r') \diamond \boxed{\text{lbl } l} \diamond \boxed{\text{lbl } r}$$

Combinatorial basis **BCKW**



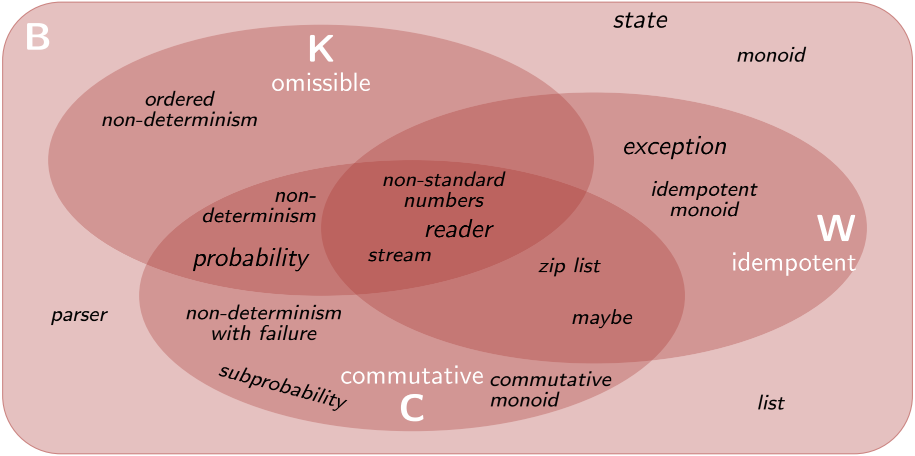
- Declarative characterisation of "liftable" equations
- Modular implementation via bracket abstraction

Combinatorial basis **BCKW**



- Declarative characterisation of “liftable” equations
- Modular implementation via bracket abstraction
- User declares and proves combinator properties at registration

Combinatorial basis **BCKW**



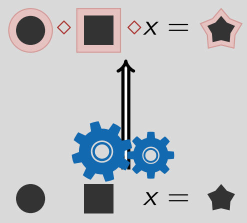
- Declarative characterisation of “liftable” equations
- Modular implementation via bracket abstraction
- User declares and proves combinator properties at registration

Summary

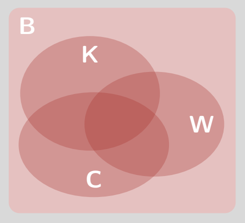
www.isa-afp.org/entries/Applicative_Lifting.shtml

```
applicative state
for
  pure: pure_state
  ap: ap_state
proof (prove)
goal (4 subgoals):
 1. ⋀f x. pure f ⋄ pure x = pure (
 2. ⋀g f x. pure (λg f x. g (f x))
 3. ⋀x. pure (λx. x) ⋄ x = x
 4. ⋀f x. f ⋄ pure x = pure (λf. f
```

functor registration



proof tactic



classify effects

guides

formalisation of the meta theory

Summary

www.isa-afp.org/entries/Applicative_Lifting.shtml