

On the Formalization of Fourier Transform in Higher-order Logic (Rough Diamond)

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ITP, 2016
Nancy, France

August 22, 2016

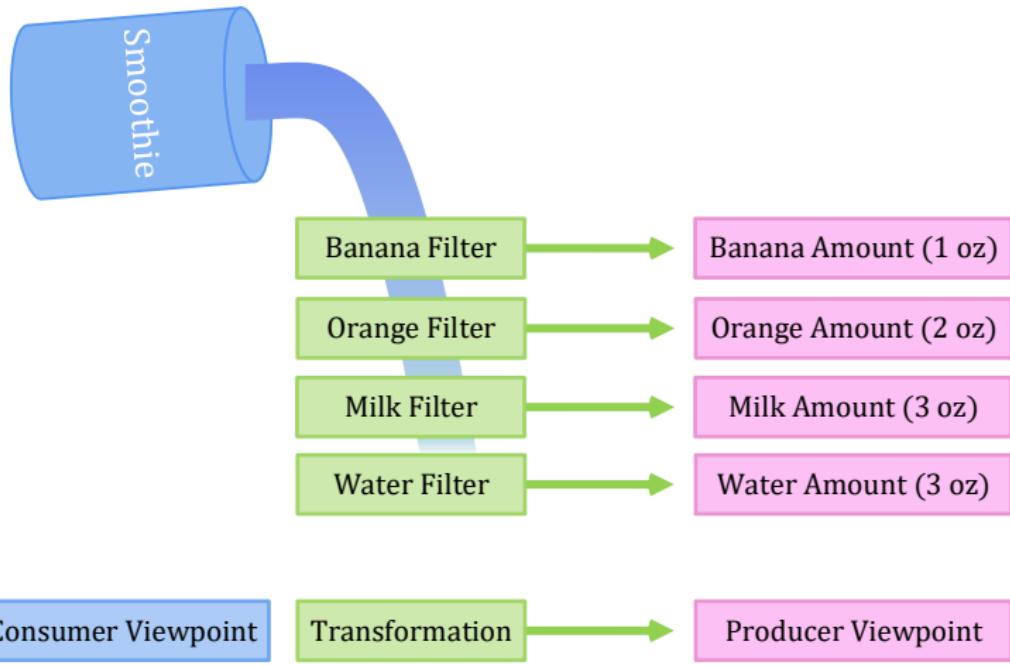


Overview

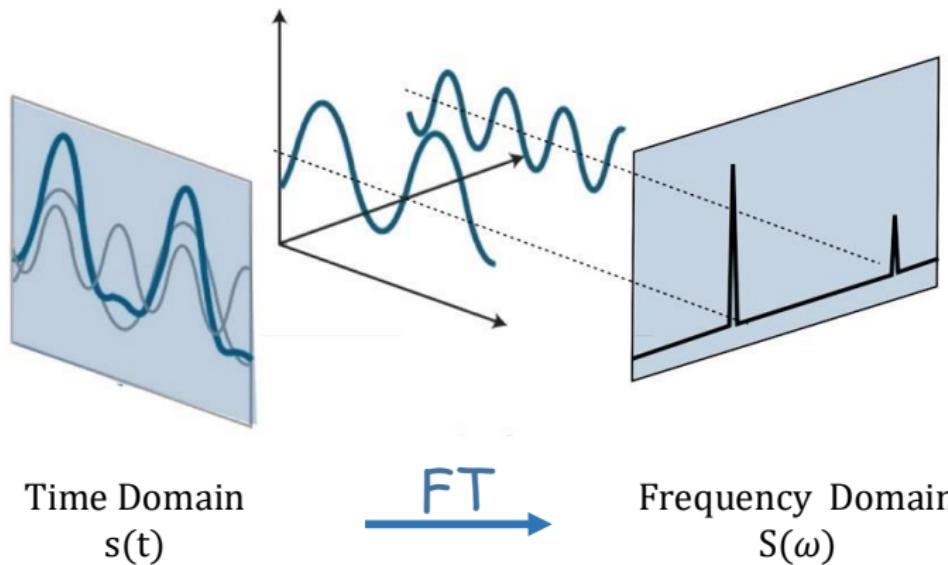
- 1 Introduction and Motivation
- 2 Formalization Details
- 3 Case Study (Automobile Suspension System)
- 4 Conclusions

Fourier Transform: Analogy

- Smoothie to Recipe Example



Fourier Transform: Graphical Representation



Time Domain
 $s(t)$

FT

Frequency Domain
 $S(\omega)$

Fourier Transform

- An **integral**-based transform method
- Mathematically, expressed as:

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt, \omega \in \mathbb{R}$$

- A linear operator
 - Accepts a **time** varying **non-causal** function $f(t)$ with real argument $t \in \mathbb{R}$
 - Outputs the corresponding ω -domain representation $F(\omega)$ with $\omega \in \mathbb{R}$

Fourier Transform: Utilization

- Solve linear Partial Differential Equations (PDEs) using simple algebraic techniques
- Used to conduct frequency response analysis of the continuous-time engineering and physical systems

Fourier Transform: Example

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Second Order Linear Partial Differential Equation

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Second Order Linear Partial Differential Equation

$$u_t = c^2 u_{xx}$$

Equivalently written as

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Taking Fourier Transform on both sides

$$U(\omega, t) = c^2(i\omega)^2 U(\omega, t) = -c^2\omega^2 U(\omega, t)$$

Using the Fourier of derivative of function and the linearity property

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$$\frac{\partial}{\partial t} U(\omega, t) = -c^2\omega^2 U(\omega, t)$$

Equivalent written as

$$U(\omega, t) = K(\omega)e^{-c^2\omega^2 t}$$

Solution in ω -domain

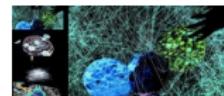
Real-world Applications for Fourier Transform

- Integral part of **analyzing** many engineering and physical systems

Signal Processing



Image Processing



Mechanical Networks



Optical Systems



Antenna Designing



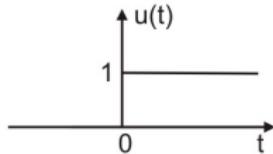
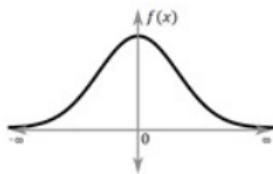
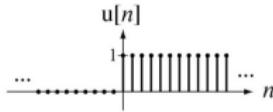
Communication Systems



Transform Methods: Related Work

- S.H. Taqdees and O. Hasan, **Formalization of Laplace Transform Using the Multivariate Calculus Theory of HOL-Light; Logic for Programming Artificial Intelligence and Reasoning (LPAR 2013)**, Lecture Notes in Computer Science Volume 8312, Springer, pp. 744-758.
- U. Siddique, M. Y. Mahmoud and S. Tahar, **On the Formalization of Z-Transform in HOL; Interactive Theorem Proving (ITP 2014)**, Lecture Notes in Computer Science Volume 8558, 2014, Springer, pp. 483-498.

Transform Methods: Comparison

| Transform Methods | Governing Equations | Input Function | Input Function Type | Output Function |
|-------------------|--|---------------------------------|---|------------------------------------|
| Laplace Transform | $F(s) = \int_0^{\infty} f(t)e^{-st} dt$ | Continuous time domain function | Semi-finite / causal function  | s-domain ($s = a + i\omega$) |
| Fourier Transform | $F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt$ | Continuous time domain function | Stable / Convergent function  | ω -domain ($s = i\omega$) |
| z-transform | $F(z) = \sum_0^{+\infty} f[n]z^{-n}$ | Discrete time function | Sampled function  | z-domain ($z = e^{is}$) |

Formal Definition of Fourier Transform

$$\text{Mathematical Definition: } \mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt, \omega \in \mathbb{R}$$

Given f is piecewise smooth and is absolutely integrable on the whole real line, i.e., f is absolutely integrable on both the positive and negative real lines

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Definition: Fourier Transform

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|- ∀ w f. fourier f w =
  integral UNIV (λt. cexp (-(ii * Cx w) * Cx (drop t))) * f t)
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```

Definition: Fourier Existence

```
|- ! f w a b. fourier_exists f =
  (! a b. f piecewise_differentiable_on interval [lift a, lift b]) ∧
  f absolutely_integrable_on {x | &0 <= drop x} ∧
  f absolutely_integrable_on {x | drop x <= &0}
```

Formalized Fourier Transform Properties

| Mathematical Form | Formalized Form |
|--|---|
| Existence of the Improper Integral of Fourier Transform | |
| $f(t)e^{-j\omega t}$ integrable on $(-\infty, +\infty)$ | $\vdash \forall f w. \text{fourier_exists } f \Rightarrow$ $(\lambda t. \text{cexp}((ii * \text{Cx } w) * \text{Cx}(\text{drop } t)) * f t)$ integrable_on UNIV |
| Linearity | |
| $\mathcal{F}[\alpha f(t) + \beta g(t)] =$ $\alpha F(\omega) + \beta G(\omega)$ | $\vdash \forall f g w a b.$ $\text{fourier_exists } f \wedge \text{fourier_exists } g \Rightarrow$ $\text{fourier}(\lambda t. a * f t + b * g t) w =$ $a * \text{fourier } f w + b * \text{fourier } g w$ |
| Frequency Shifting | |
| $\mathcal{F}[e^{j\omega_0 t} f(t)] = F(\omega - \omega_0)$ | $\vdash \forall f w w_0.$ $\text{fourier_exists } f \Rightarrow$ $\text{fourier}(\lambda t. \text{cexp}((ii * \text{Cx } (w_0)) * \text{Cx}(\text{drop } t)) * f t) w =$ $\text{fourier } f (w - w_0)$ |
| Modulation | |
| $\mathcal{F}[\cos(\omega_0 t) f(t)] =$ $\frac{F(\omega - \omega_0) + F(\omega + \omega_0)}{2}$ | $\vdash \forall f w w_0.$ $\text{fourier_exists } f \Rightarrow$ $\text{fourier}(\lambda t. \text{ccos}(\text{Cx } w_0 * \text{Cx}(\text{drop } t)) * f t) w =$ $(\text{fourier } f (w - w_0) + \text{fourier } f (w + w_0)) / \text{Cx } (\&2)$ |
| $\mathcal{F}[\sin(\omega_0 t) f(t)] =$ $\frac{F(\omega - \omega_0) - F(\omega + \omega_0)}{2i}$ | $\vdash \forall f w w_0.$ $\text{fourier_exists } f \Rightarrow$ $\text{fourier}(\lambda t. \text{csin}(\text{Cx } w_0 * \text{Cx}(\text{drop } t)) * f t) w =$ $(\text{fourier } f (w - w_0) - \text{fourier } f (w + w_0)) / (\text{Cx } (\&2) * ii)$ |

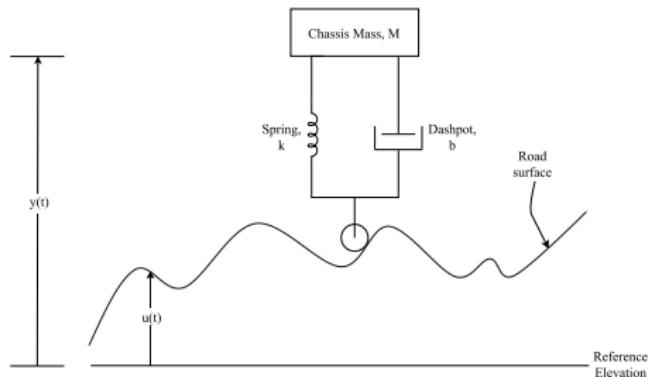
Formalized Fourier Transform Properties

| Mathematical Form | Formalized Form |
|---|---|
| | Time Reversal |
| $\mathcal{F}[f(-t)] = F(-\omega)$ | $\vdash \forall f w. \text{fourier_exists } f \Rightarrow$ $\quad \text{fourier } (\lambda t. f (-t)) w = \text{fourier } f (-w)$ |
| | First-order Differentiation |
| $\mathcal{F}\left[\frac{d}{dt} f(t)\right] = i\omega F(\omega)$ | $\vdash \forall f w.$ $\quad \text{fourier_exists } f \wedge$ $\quad \text{fourier_exists } (\lambda t. \text{vector_derivative } f (\text{at } t)) \wedge$ $\quad (\forall t. f \text{ differentiable at } t) \wedge$ $\quad ((\lambda t. f (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_posinfinity} \wedge$ $\quad ((\lambda t. f (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_neginfinity}$ $\Rightarrow \quad \text{fourier } (\lambda t. \text{vector_derivative } f (\text{at } t)) w =$ $\quad \quad \quad \text{ii} * \text{Cx } w * \text{fourier } f w$ |
| | Higher-order Differentiation |
| $\mathcal{F}\left[\frac{d^n}{dt^n} f(t)\right] = (i\omega)^n F(\omega)$ | $\vdash \forall f w n.$ $\quad \text{fourier_exists_higher_deriv } n f \wedge$ $\quad (\forall t. \text{differentiable_higher_derivative } n f t) \wedge$ $\quad (\forall p. p < n \Rightarrow$ $\quad \quad ((\lambda t. \text{higher_vector_derivative } p f (\text{lift } t)) \rightarrow \text{vec } 0)$ $\quad \quad \quad \text{at_posinfinity}) \wedge$ $\quad (\forall p. p < n \Rightarrow$ $\quad \quad ((\lambda t. \text{higher_vector_derivative } p f (\text{lift } t)) \rightarrow \text{vec } 0)$ $\quad \quad \quad \text{at_neginfinity})$ $\Rightarrow \quad \text{fourier } (\lambda t. \text{higher_vector_derivative } n f t) w =$ $\quad \quad \quad (\text{ii} * \text{Cx } w) \text{ pow } n * \text{fourier } f w$ |

- 3500 lines of HOL-Light code and approximately 300 man-hours

Case Study: Automobile Suspension System (ASS)

- A mechanical system
- Is intended to filter out the rapid variations on the road surface
 - To act as a low-pass filter



Differential Equation:

$$M \frac{d^2y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = ku(t) + b \frac{du(t)}{dt}$$

Frequency Response:

$$\frac{Y(\omega)}{U(\omega)} = \frac{\frac{b}{M}(i\omega) + \frac{k}{M}}{(i\omega)^2 + \frac{b}{M}(i\omega) + \frac{k}{M}}$$

Automobile Suspension System (ASS)

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Definition: Differential Equation of order n

```
|- ! n L f t. differential_equation n L f t <=>
  vsum (0..n) (\k. EL k L t * higher_order_derivative k f t )
```

Automobile Suspension System (ASS)

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```

Definition: Differential Equation of ASS

```
|- ∀ y u a b c. diff_eq_ASS y u M b k ⇔  
  ( ∀t. differential_equation 2 [Cx k; Cx b ; Cx M] y t =  
    differential_equation 1 [Cx k; Cx b] u t )
```

Automobile Suspension System (ASS)

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$$\frac{Y(\omega)}{U(\omega)} = \frac{\frac{b}{M}(i\omega) + \frac{k}{M}}{(i\omega)^2 + \frac{b}{M}(i\omega) + \frac{k}{M}}$$

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Automobile Suspension System (ASS)

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Automobile Suspension System (ASS)

Frequency Response:

$$\frac{Y(\omega)}{U(\omega)} = \frac{\frac{b}{M}(i\omega) + \frac{k}{M}}{(i\omega)^2 + \frac{b}{M}(i\omega) + \frac{k}{M}}$$

Theorem: Frequency Response of ASS

$\vdash \forall y u w a. \quad &0 < M \wedge &0 < b \wedge &0 < k \wedge$
 $(\forall t. \text{differentiable_higher_derivative } 2 y t) \wedge$
 $(\forall t. \text{differentiable_higher_derivative } 1 u t) \wedge$
 $\text{fourier_exists_higher_deriv } 2 y \wedge \text{fourier_exists_higher_deriv } 1 u \wedge$
 $(\forall p. \quad p < 2 \Rightarrow$
 $((\lambda t. \text{higher_vector_derivative } p y (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_posinfinity} \wedge$
 $(\forall p. \quad p < 2 \Rightarrow$
 $((\lambda t. \text{higher_vector_derivative } p y (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_neginfinity} \wedge$
 $((\lambda t. \quad u (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_posinfinity} \wedge$
 $((\lambda t. \quad u (\text{lift } t)) \rightarrow \text{vec } 0) \text{ at_neginfinity} \wedge$
 $(\forall t. \quad \text{diff_eq_ASS } y u a b c) \wedge \sim(\text{fourier } u w = \text{Cx } (&0)) \wedge$
 $\sim((ii * \text{Cx } w) \text{ pow } 2 + \text{Cx } (b / M) * ii * \text{Cx } w + \text{Cx } (k / M) = \text{Cx } (&0))$
 $\Rightarrow (\text{fourier } y w / \text{fourier } u w = (\text{Cx } (b / M) * ii * \text{Cx } w + \text{Cx } (k / M)) /$
 $((ii * \text{Cx } w) \text{ pow } 2 + \text{Cx } (b / M) * ii * \text{Cx } w + \text{Cx } (k / M))$

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Frequency Response:

$$\frac{Y(\omega)}{U(\omega)} = \frac{\frac{b}{M}(i\omega) + \frac{k}{M}}{(i\omega)^2 + \frac{b}{M}(i\omega) + \frac{k}{M}}$$

Theorem: Frequency Response of ASS

```
|- ! y u w a. &0 < M ∧ &0 < b ∧ &0 < k ∧
  (!t. differentiable_higher_derivative 2 y t) ∧
  (!t. differentiable_higher_derivative 1 u t) ∧
  fourier_exists_higher_deriv 2 y ∧ fourier_exists_higher_deriv 1 u ∧
  (!p. p < 2 ⇒
    ((!t. higher_vector_derivative p y (lift t)) → vec 0) at_posinfinity) ∧
  (!p. p < 2 ⇒
    ((!t. higher_vector_derivative p y (lift t)) → vec 0) at_neginfinity) ∧
  ((!t. u (lift t)) → vec 0) at_posinfinity ∧
  ((!t. u (lift t)) → vec 0) at_neginfinity ∧
  (!t. diff_eq_ASS y u a b c) ∧ ~ (fourier u w = Cx (&0)) ∧
  ~ ((ii * Cx w) pow 2 + Cx (b / M) * ii * Cx w + Cx (k / M) = Cx (&0))
  ⇒ (fourier y w / fourier u w = (Cx (b / M) * ii * Cx w + Cx (k / M)) /
      ((ii * Cx w) pow 2 + Cx (b / M) * ii * Cx w + Cx (k / M)))
```

- 500 lines of HOL-Light code and the proof process took just a couple of hours

Conclusions

- Formalization of Fourier transform using higher-order logic
- Foundations Utilized
 - Multivariate Calculus theory of HOL-Light
 - Integration
 - Differential
 - Transcendental
 - Topological
- Case Study
 - Automobile Suspension System

Future Work

- Formalization of **inverse** Fourier transform
 - Whole framework will provide **solutions** to **differential equations**
- Formalization of **2-dimensional** Fourier transform
 - Widely used for the analysis
 - Optical systems
 - Electromagnetics
 - Image processing

Thanks!

- For More Information

- Visit our website

- <http://save.seecs.nust.edu.pk/>

- Contact

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