HF Sets in Constructive Type Theory

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A minimal computational axiomatization
of HF sets
with a unique model.

What are Hereditarily Finite sets?

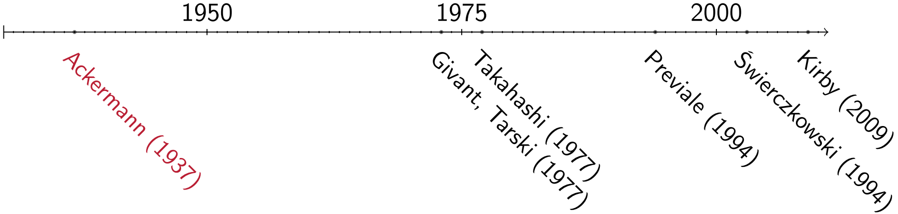
= all finite, well-founded sets whose elements are HF again

What are HF sets useful for?

Świerczkowski (1994), Paulson (2015)

Previous Work

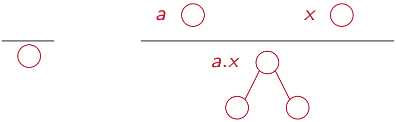
$hf \approx \mathbb{N}$

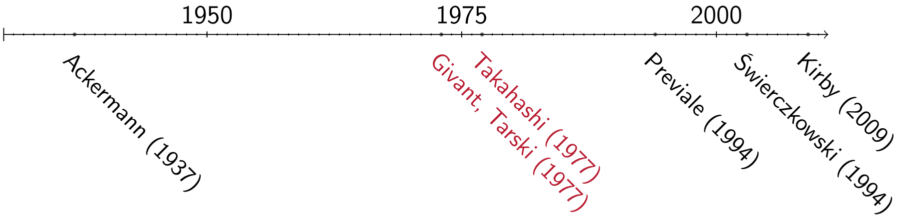


$$\begin{array}{ccccccc} & & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 21 & \approx & 1 & 0 & 1 & 0 & 1 \\ \lceil 21 \rceil & = & \{\lceil 4 \rceil, & & \lceil 2 \rceil, & & \lceil 0 \rceil\} \end{array}$$

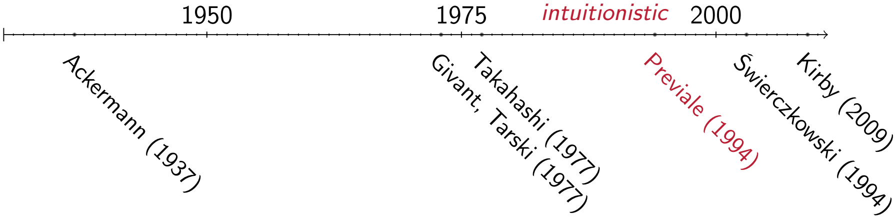
Previous Work

$$\frac{}{\emptyset:\mathrm{HF}} \qquad \frac{x:\mathrm{HF} \qquad y:\mathrm{HF}}{\underbrace{\{x\}\cup y}_{x.y}:\mathrm{HF}}$$

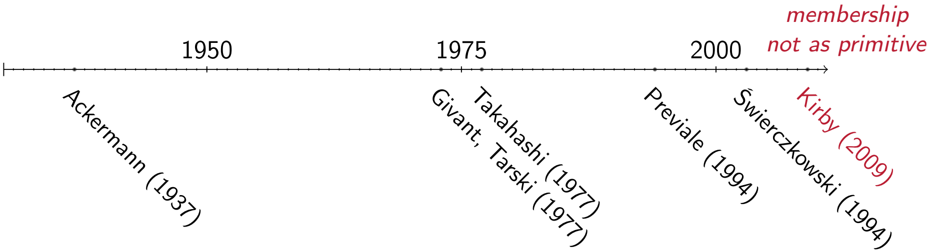




Previous Work



Previous Work



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What is needed for HF sets?

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1. Constants: hf, $\emptyset$, $a.x$

   $x \in y := x.y = y$

2. A characterization of equality

$$
\begin{array}{lcll}
x.x.y & = & x.y & (\mathit{cancellation}) \\
x.y.z & = & y.x.z & (\mathit{swap}) \\
x.y & \neq & \emptyset & (\mathit{discreteness}) \\
\underbrace{x.y.z = y.z \rightarrow x = y \vee x.z = z}_{x \in y.z \rightarrow x = y \,\vee\, x \in z} & & & (\mathit{membership})
\end{array}
$$

3. A strong induction principle

$$
\begin{array}{c}
\forall p : \mathrm{hf} \rightarrow \mathrm{Type}.\ p\ \emptyset \\
\rightarrow (\forall x\ y.\ p\ x \rightarrow p\ y \rightarrow p\ (x.y)) \rightarrow \forall x.\ p\ x
\end{array}
$$

Working with the Induction Principle

$$R: \; p\,\emptyset \rightarrow (\forall x\, y.\; p\, x \rightarrow p\, y \rightarrow p\,(x.y)) \rightarrow \forall x.\; p\, x$$

$$R\; p_0\; p_S\; \emptyset \overset{?}{=} p_0$$

$$R\; p_0\; p_S\; (a.x) \overset{?}{=} p_S\; (R\; p_0\; p_S\; a)\; (R\; p_0\; p_S\; x)$$

$$\begin{array}{lcl} \pi_1\; \emptyset & = & \text{None} \\ \pi_1\; (a.x) & = & \text{Some}\; a \end{array} \quad ↯$$

Working with the Induction Principle

$$R : \; p\,\emptyset \rightarrow (\forall x\,y.\; p\,x \rightarrow p\,y \rightarrow p\,(x.y)) \rightarrow \forall x.\; p\,x$$

1 Recursive Specification

$$\begin{array}{lcccl} \emptyset & \cup & y & = & y \\ a.x & \cup & y & = & a.(x \cup y) \end{array}$$

2 Membership Specification

$$z \in x \cup y \leftrightarrow z \in x \vee z \in y$$

1 Membership Specification

$$\Sigma u.\; \forall z.\; z \in u$$
$$\leftrightarrow z \in x \vee z \in y$$

2 Recursive Specification

Needed: extensionality

What is not needed as primitives?

1. Membership
   $x \in y := x.y = y$
2. Recursion equations
3. Decidability of equality: dep. on extensionality
4. Extensionality: dep. on decidability of equality

Extensionality and Decidability Results

$dec\ (x \in \underline{y})$

$dec\ (y \in \underline{x})$

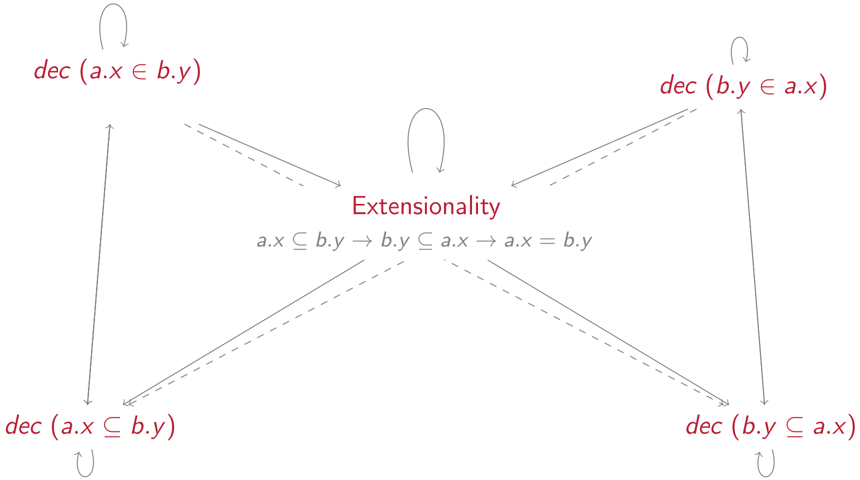
Extensionality

$x \subseteq y \to y \subseteq x \to \underline{x} = \underline{y}$

$dec\ (\underline{x} \subseteq y)$

$dec\ (\underline{y} \subseteq x)$

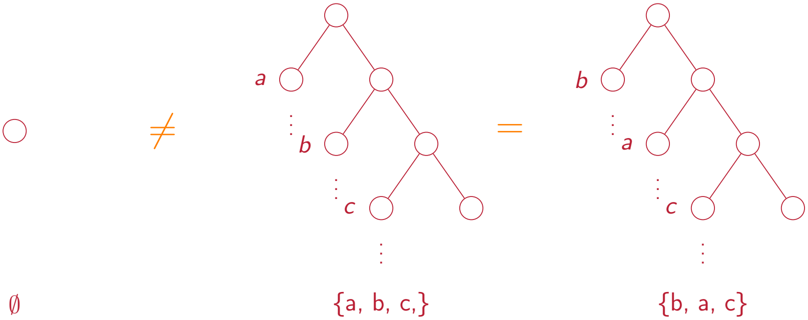
Extensionality and Decidability Results



A minimal computational axiomatization of HF sets with a unique model.

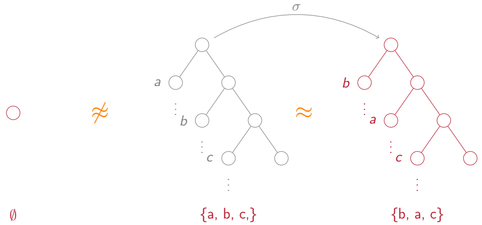
existence uniqueness

A Tree Model for HF Sets



HF sets $= \emptyset + a.x +$ equality $+$ induction principle

A Tree Model for HF Sets



1. An inductive type representing the tree structure:

$$T := 0 \mid T \,.\, T$$

2. An equivalence relation $\approx : T \to T \to \mathrm{Prop}$
3. An idempotent normalizer $\sigma : T \to T$ s.t.

$$s \approx t \leftrightarrow \sigma s = \sigma t$$

4. Construct a subtype X of T only containing normalized trees.

Definition of $\approx$

Equivalence

$$\frac{}{s.s.t \approx s.t} \qquad \frac{}{s.t.u \approx t.s.u}$$

$$\frac{}{s \approx s} \qquad \frac{s \approx t}{t \approx s} \qquad \frac{s \approx t \quad t \approx u}{s \approx u} \qquad \frac{s \approx s' \quad t \approx t'}{s.t \approx s'.t'}$$

To show: $\approx$ satisfies the equality axioms of HFs, for example

1. $s.s.t \;\approx\; s.t$
2. $s.t.u \;\approx\; t.u \rightarrow s \;\approx\; t \vee s.u \approx u$

A Normalization Function

Idea: Use sorted trees as normal form.

Lexical Tree Order

$$\frac{}{0 < s.t} \qquad \frac{s < s'}{s.t < s'.t'} \qquad \frac{t < t'}{s.t < s.t'}$$

Define a sort function $\sigma : T \to T$ according to the above order satisfying

1. $\sigma\ (\sigma\ s) = \sigma\ s$
2. $s\ \approx\ t \leftrightarrow \sigma s = \sigma t$

$\Rightarrow$ There exists a type $\{t \mid \sigma t = t\}$.

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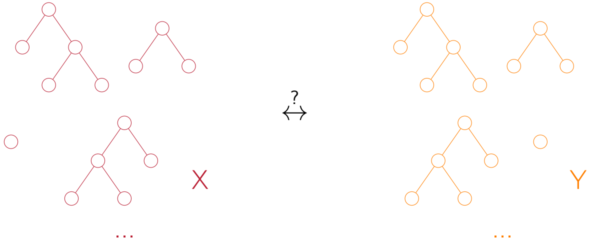
existence uniqueness

Are all HF structures the same?



$f : X \to Y$
homomorphism:

$$\begin{aligned} f\ \emptyset &= \emptyset \\ f\ (a.x) &= (f\ a).(f\ x) \end{aligned}$$

Are all HF structures the same?

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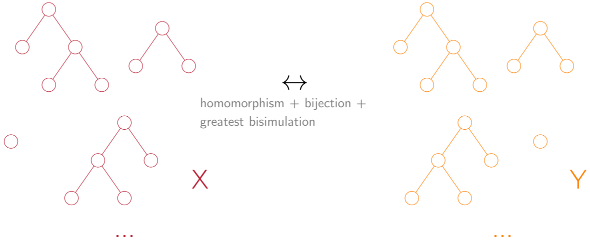

$$\frac{}{R\ \emptyset\ \emptyset} \qquad \frac{R\ a\ b \quad R\ x\ y}{R\ a.x\ b.y}$$

1. **Totality** $\forall x.\ \Sigma y.\ R\ x\ y.$
2. **Functionality** $R\ x\ y \to R\ x\ y' \to y = y'$
   - **Simulation** $R\ x\ y \to a \in x \to \exists\ b. b \in y \land R\ a\ b$
3. f homomorphism $\Rightarrow\ R\ x\ (f\ x)$
4. All homomorphisms between HF structures are equivalent.
5. All HF structures are isomorphic.

A minimal computational axiomatization
of HF sets
with a unique model.

Axiomatization + Discreteness +
Operations + Ordinals + Categoricity +
Model Construction

Everything is formalized in Coq.

$\sim$ 2000 lines

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Everything is formalized in Coq.

similar to proofs in paper

special-purpose tactic based on intro-elim rules

Everything is formalized in Coq.

no inductive types except for the model construction

Everything is formalized in Coq.

Where? - www.ps.uni-saarland.de/extras/hfs

Contribution

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- First minimal, computationally complete axiomatization of HF sets
- Operationally complete axiomatization
- First proof of categoricity

Further Work

- A recursor with equations
- Axiomatization of non-wellfounded sets

Thank you for your attention!

Where? - www.ps.uni-saarland.de/extras/hfs