Computational Logic

Algebraic Numbers in Isabelle/HOL[1]

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Overview

- Motivation
- Real Algebraic Numbers
  - Well-Definedness
  - Calculating Real Roots of Rational Polynomial
  - Factorizing Rational Polynomial
  - Arithmetic on Real Algebraic Numbers
- Complex Algebraic Numbers

Certify Complexity of Matrix Interpretations

- given: automatically generated complexity proof for program

$$\chi_A = (x - 1) \cdot (-39 + 360x - 832x^2 + 512x^3)$$

- criterions
  - polynomial complexity if norms of all complex roots of $\chi_A \leqslant 1$
  - degree d in $\mathcal{O}(n^d)$: more calculations with complex roots of χ_A

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  - degree d in $\mathcal{O}(n^d)$: more calculations with complex roots of χ_A
- problem: certifier crashed as numbers got too complicated

Closed Form for Cubic Polynomials

- $-39 + 360x - 832x^2 + 512x^3 = 0$ iff
  - $x = \frac{1}{24}\left(13 + \frac{34}{\sqrt[3]{91+9i\sqrt{383}}} + \sqrt[3]{91+9i\sqrt{383}}\right)$,
  - $x = \frac{13}{24} - \frac{17\left(1+i\sqrt{3}\right)}{24\sqrt[3]{91+9i\sqrt{383}}} - \frac{1}{48}\left(1 - i\sqrt{3}\right)\sqrt[3]{91+9i\sqrt{383}}$, or
  - $x = \frac{13}{24} - \frac{17\left(1-i\sqrt{3}\right)}{24\sqrt[3]{91+9i\sqrt{383}}} - \frac{1}{48}\left(1 + i\sqrt{3}\right)\sqrt[3]{91+9i\sqrt{383}}$

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for all $j \in \{1, 2, 3\}$

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  for all $j \in \{1, 2, 3\}$
- problem: no closed form for roots of polynomials of degree 5 and higher

Algebraic Numbers

- number $x \in \mathbb{R} \cup \mathbb{C}$ is algebraic iff it is root of non-zero rational polynomial
  - $x_1 =$ "root #1 of $-39 + 360x - 832x^2 + 512x^3$"
  - $x_2 =$ "root #2 of $-39 + 360x - 832x^2 + 512x^3$"
  - $x_3 =$ "root #3 of $-39 + 360x - 832x^2 + 512x^3$"

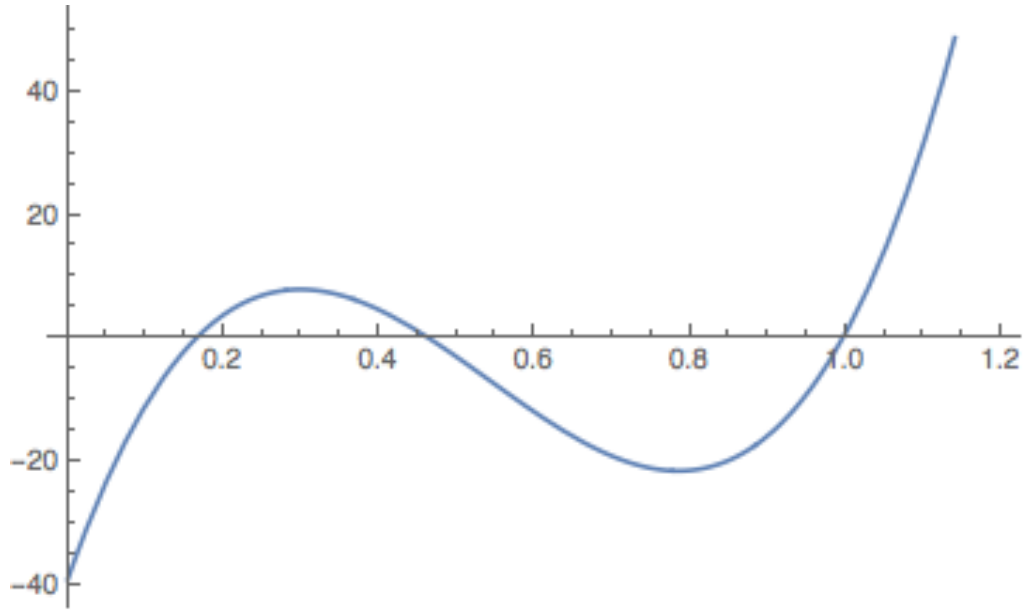

Figure: $-39 + 360x - 832x^2 + 512x^3$

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Problems with Algebraic Numbers

- well-definedness — Sturm's method

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- comparisons — Sturm's method

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- arithmetic — matrices, determinants, resultants, ...

  calculate a polynomial representing

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Main Result: Formalization of Algebraic Numbers

- common properties on algebraic numbers ($\mathbb{R}$ and $\mathbb{C}$)
- executable real algebraic numbers
- executable complex algebraic numbers indirectly
- easy to use via data-refinement for $\mathbb{R}$ and $\mathbb{C}$

$$\lfloor \mathit{norm}(\sqrt{\sqrt[3]{2}+3+2i}) \cdot 100 \rfloor \overset{\mathit{evaluate}}{\hookrightarrow} 216$$

- applicable inside (eval) and outside Isabelle (export-code)

Related work

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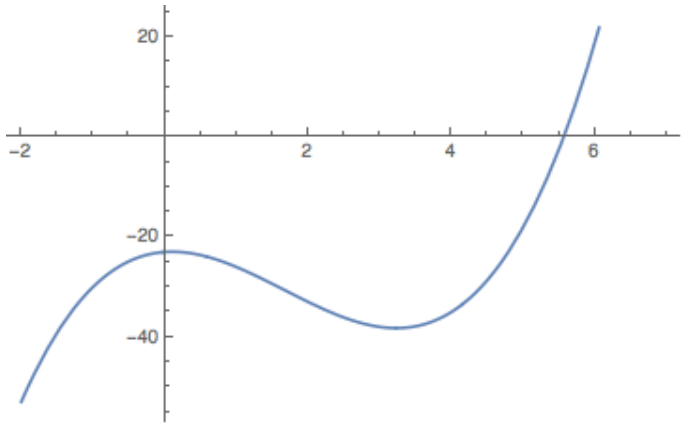
- experimental comparison (examples of Li and Paulson)

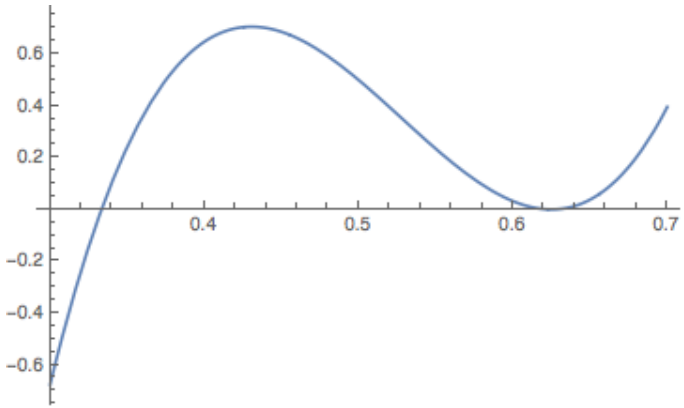
MetiTarski	1.83 seconds	(@ 2.66 Ghz)
+ validation of Li and Paulson	4.16 seconds	(@ 2.66 Ghz)
Our generated Haskell code	0.03 seconds	(@ 3.5 Ghz)

Well-Definedness

- "root #3 of $-23 + x - 5x^2 + x^3$"



- "root #3 of $-25 + 155x - 304x^2 + 192x^3$"



Sturm's Method

- input
  - polynomial over $\mathbb{R}$
  - interval ($[2, 5]$, $(-\pi, 7]$, $(-\infty, 3)$, or $(-\infty, +\infty)$)
- output: count-roots *p itval*
  - number of distinct real roots of *p* in interval *itval*

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  - adapted functions to work over $\mathbb{Q}$

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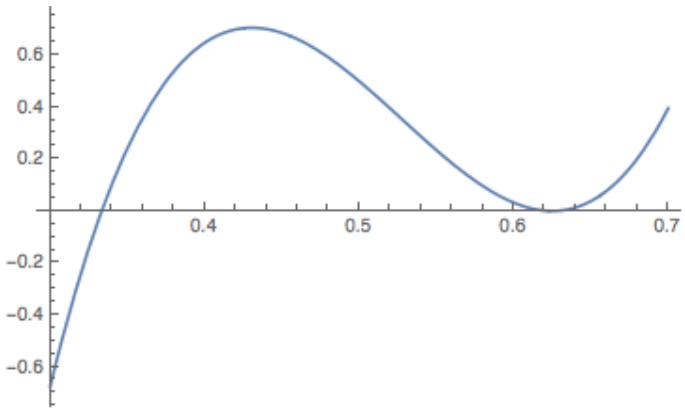
    reason: apply Sturm's method to implement $\mathbb{R}$
    formalization: locale for homomorphisms
  - precompute first phase of Sturm's method

  $\Rightarrow$ query many intervals for same polynomial more efficiently

Applying Sturm's Method for Well-Definedness

"root #3 of $-25 + 155x - 304x^2 + 192x^3$"



count-roots $(-25 + 155x - 304x^2 + 192x^3)\ (-\infty, +\infty) = 2$

Representation of Real Algebraic Numbers – Part 1

- quadruple
  - polynomial over $\mathbb{Q}$: p
  - left and right interval bound $\in \mathbb{Q}$: l and r
  - precomputation of Sturm for p: root-info
  - (flag for factorization information on p)

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  - ...

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- five invariants
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  - ...
- invariants are ensured via a subtype (typedef)
  - new type of quadruples satisfying invariants
  - lift-definition and transfer for function definitions and proofs

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4. perform bisection on $[l_i, r_i]$ to obtain intervals which all contain a single root of p_i

Example

- input: χ_A
- computation:
  - $39 - 399x + 1192x^2 - 1344x^3 + 512x^4$
- output:

Example

- factorization
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  - $(x - 1) \cdot (-39 + 360x - 832x^2 + 512x^3)$
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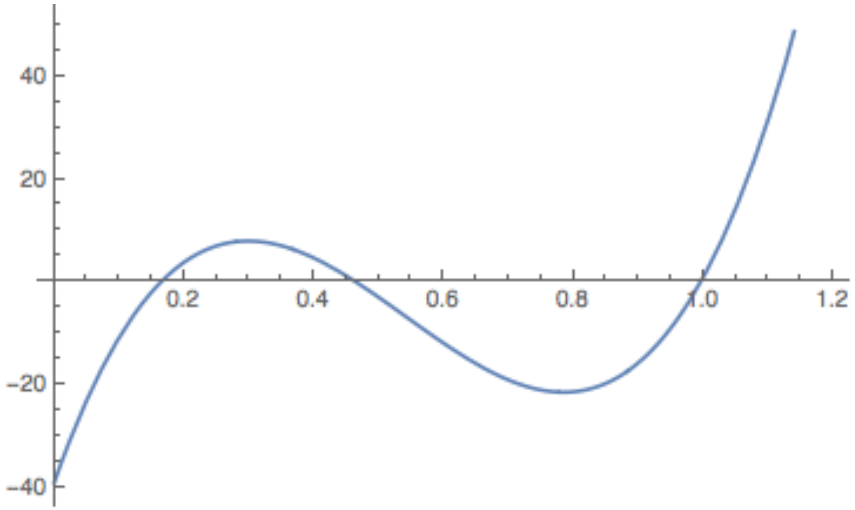
- output: $(x - 1, [1, 1])$

Example

- initial bounds: $6 = \mathit{degree}\ p_i \cdot \mathit{max}\{\lceil |c_i| \rceil .\ c_i\ \mathit{coefficient\ of}\ p_i\}$
- computation:
  - $(x-1)\cdot(-39+360x-832x^2+512x^3)$
  - $\mathit{todo} = \{[-6,6]\}$
  - 

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Example

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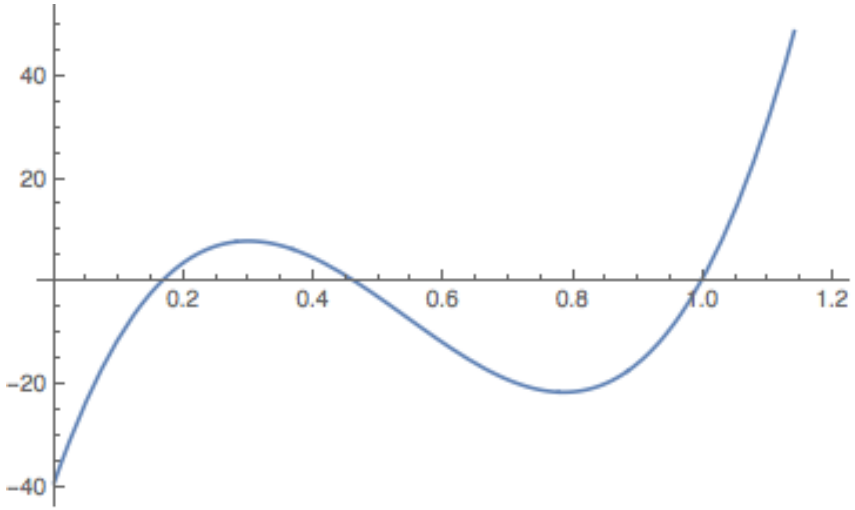
  - 

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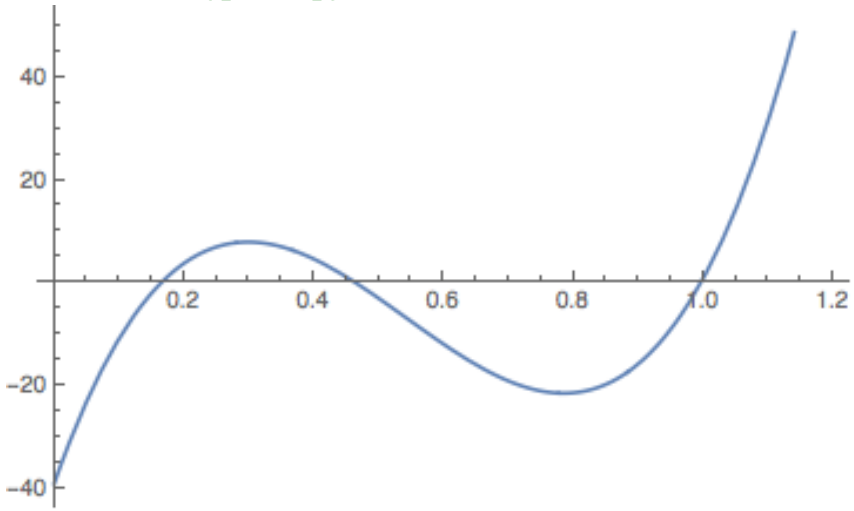
  - 

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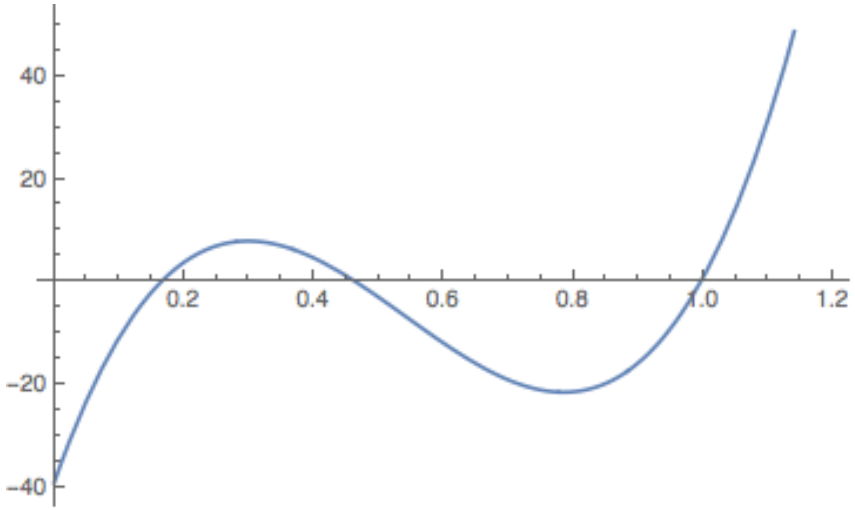
  - 

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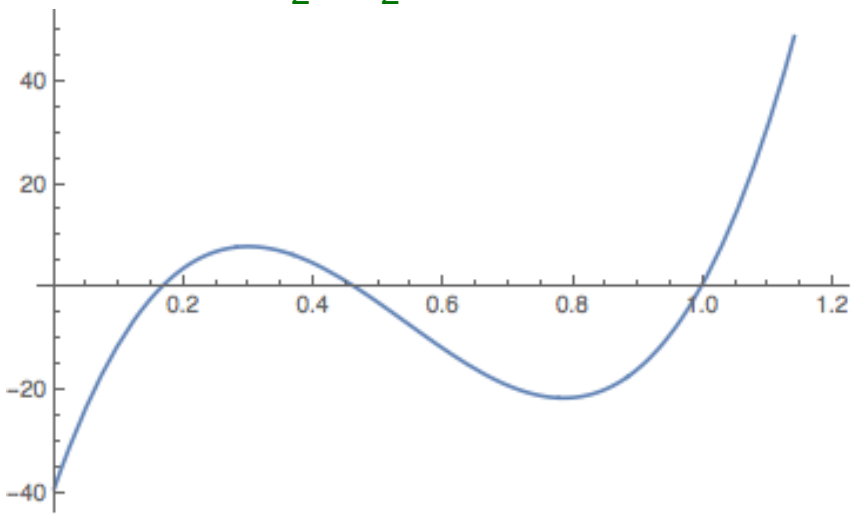
  - 

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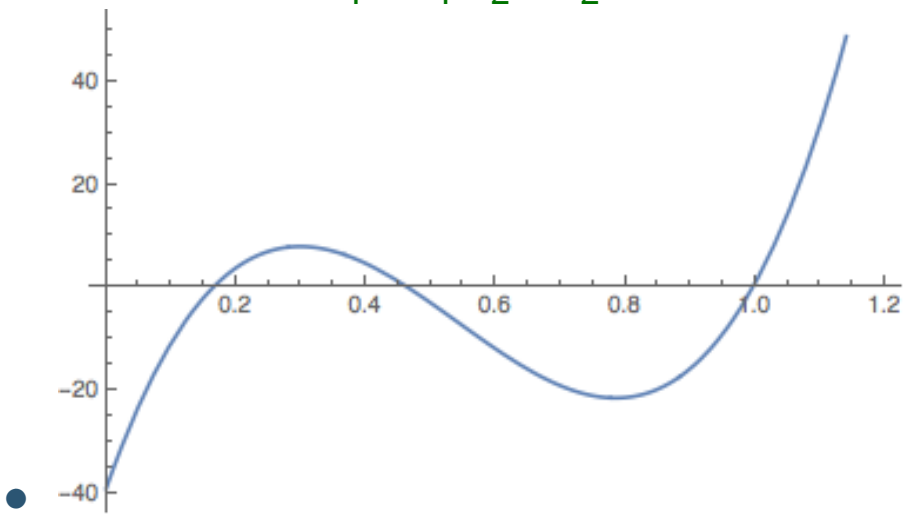
  - 

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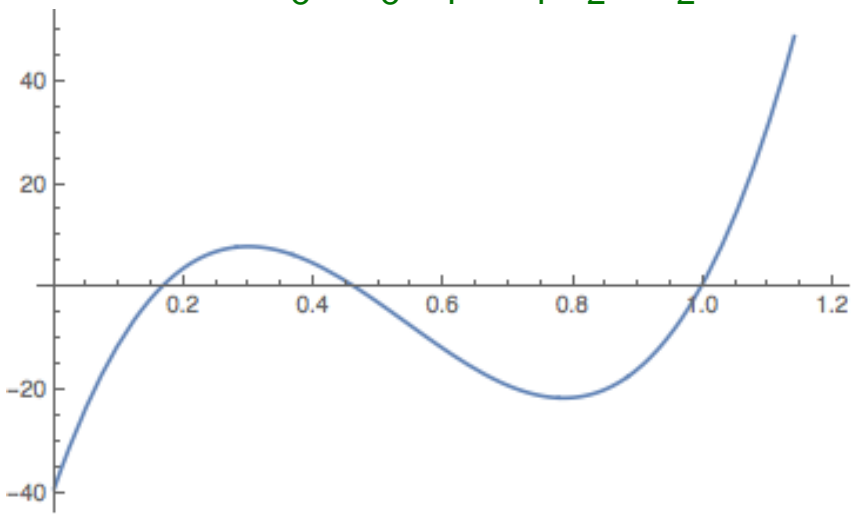
  - 

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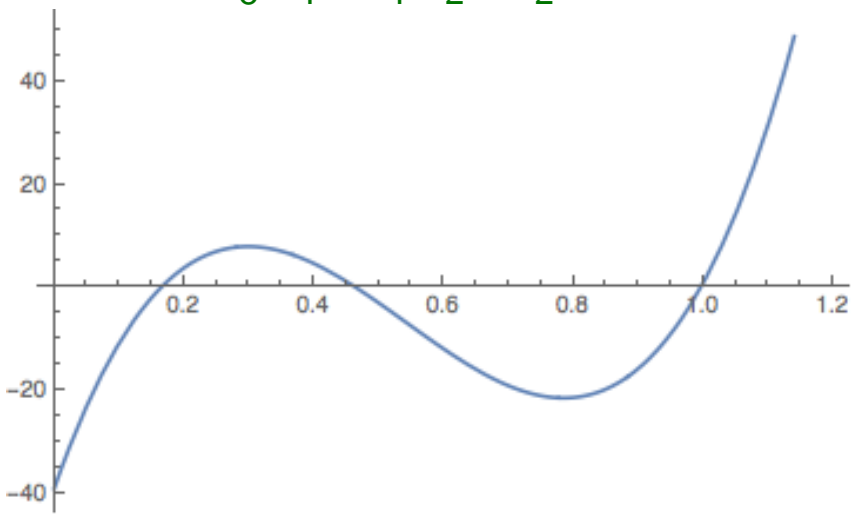
  - 

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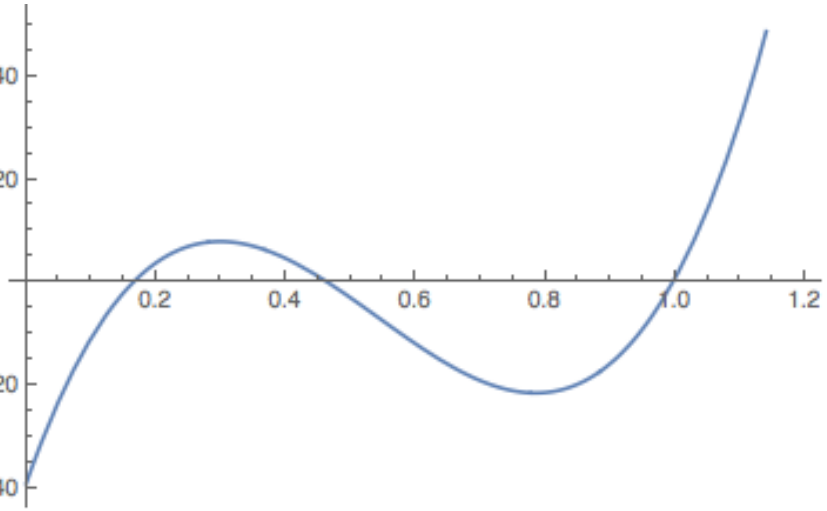
  - 

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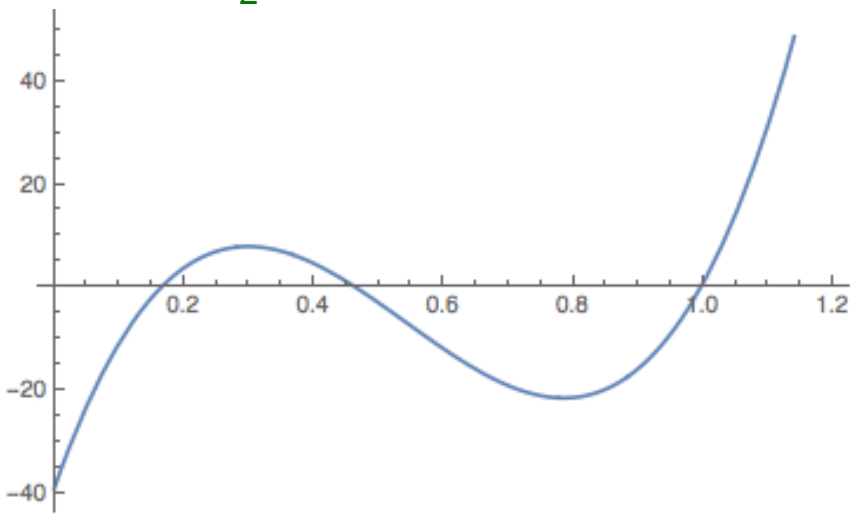
  - 

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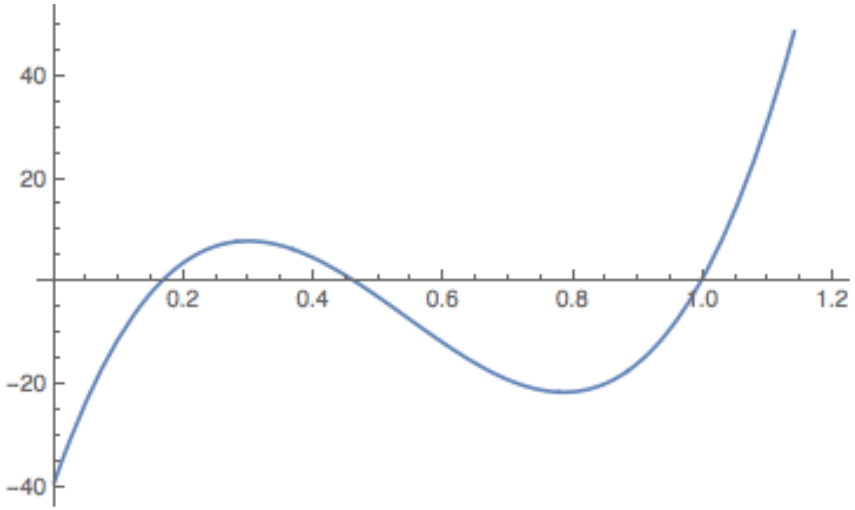
  - 

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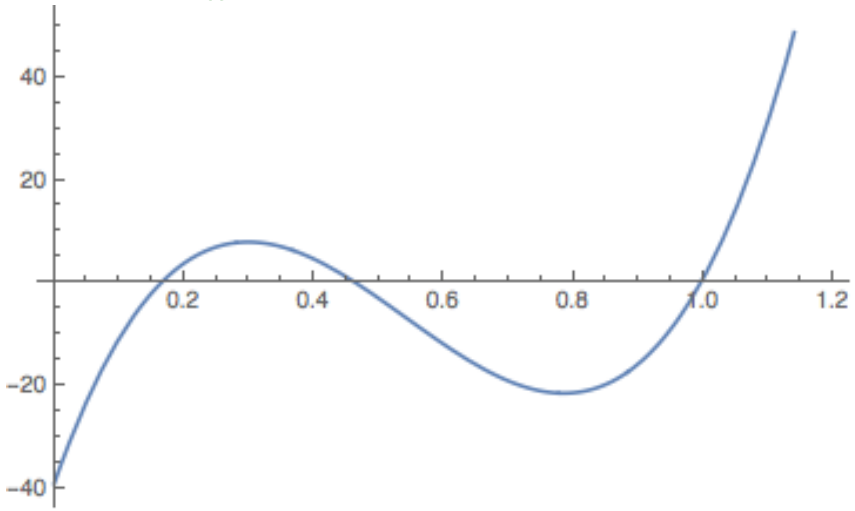
  - 

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- soundness via well-founded induction;
  order based on minimal distance between roots of p_i

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factorization	$n = 8$	$n = 9$	$n = 10$
none	2m11s/256	22m19s/512	12h19m/1024
square-free	2m14s/256	15m31s/384	9h31m/768
full	0.35s/16	0.35s/16	0.59s/16

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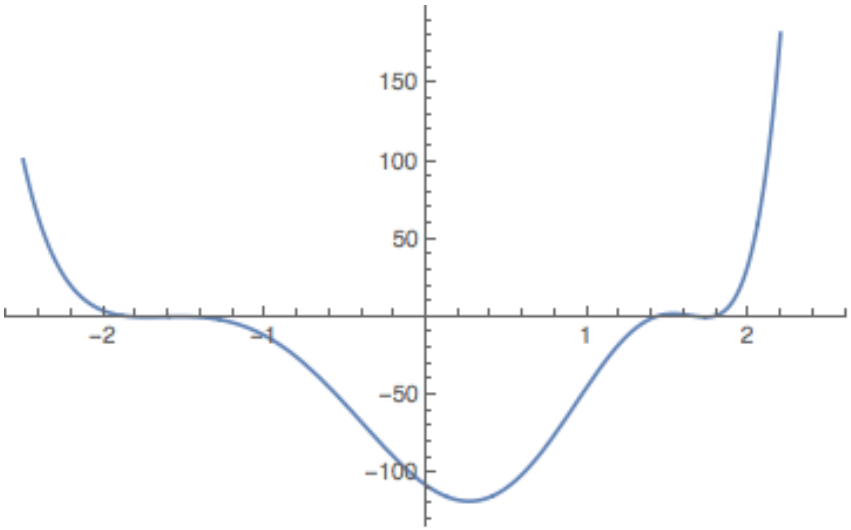
  - check equality at runtime, not irreducibility

Simplification of Representation

Consider

$p(x) = -108 - 72x + 108x^2 + 84x^3 - 27x^4 - 32x^5 - 2x^6 + 4x^7 + x^8$



$$\begin{aligned} & \text{THE root of } p \text{ in } [1, 1.5] \\ = \ & \text{THE root of } (-3 + x^2) \cdot (6 + 4x + x^2) \cdot (-2 + x^2) \text{ in } [1, 1.5] \\ = \ & \text{THE root of } -2 + x^2 \text{ in } [1, 1.5] \\ (= \ & \sqrt{2}) \end{aligned}$$

Comparison of Real Algebraic Numbers

1. decide whether $(p, [l, r])$ and $(q, [l', r'])$ encode same number

$$\text{THE root of } p \text{ in } [l, r] = \text{THE root of } q \text{ in } [l', r']$$
$$\Longleftrightarrow \textit{gcd } p\ q \text{ has real root in } [l, r] \cap [l', r']$$

2. if not, tighten bounds of both numbers via bisection until intervals are disjoint
   (bisection algorithm again defined by partial-function)

Arithmetic on Real Algebraic Numbers

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4. optimize representation

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- efficient (not optimal) computation of resultants and GCDs
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```
datatype real_alg_3 =
    Rational rat
  | Irrational "quadruple with invariants"

typedef real_alg_4 = "real_alg_3 with invariant"

definition real_of_4 :: real_alg_4 => real

quotient_type real_alg =
  real_alg_4 / "% x y. real_of_4 x = real_of_4 y"
```

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  $\Rightarrow$ compute all these roots

$$Re(x) = \left(SOME\ x.\ 8 - 24x - 88x^3 + 96x^4 + 288x^5 + 384x^6 + 768x^7 + 512x^9 = 0\right)$$

$$Im(x) = \left(SOME\ x.\ \frac{961}{4096}x^2 - \frac{279}{512}x^4 + \frac{453}{256}x^6 - \frac{85}{32}x^8 + \frac{27}{8}x^{10} - 3x^{12} + x^{14} = 0\right)$$

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and filter: for all candidates $z = Re(x) + Im(x)i$, test $p(z) = 0$

Summary

- formalization of real and complex algebraic numbers, implementing $\mathbb{R}$ and $\mathbb{C}$
  $+$, $-$, $\times$, $/$, $\sqrt[n]{\cdot}$, $\lfloor\cdot\rfloor$, $=$, $<$, *i*, *Re*, *Im*, show
- factorization algorithms for rational polynomials over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- heavily relying on Sturm's method and matrix library
- based on factorization oracle (S. Joosten @ Isabelle workshop)
- $\sim$ 20.000 loc, available in archive of formal proofs