AUTO2, a saturation-based heuristic prover for higher-order logic

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- Despite enormous progress, computers still cannot prove many statements that humans consider to be routine.
- Obstacle to both the QED project (formalization of mathematics) and more widespread adoption of formal software verification.

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  - Work with higher order logic and (simple) types.

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  - Have a robust search mechanism.
- auto2 is *not* designed to:
  - Be fully automatic.
  - Have good completeness properties.

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Overall architecture

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- **Important:** Previously proved theorems are *not* among the assumptions.
  - Instead they are encoded into set of allowed actions.

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- Best-first search: each item is assigned a **score**. New items are put into a priority queue, and are inserted into the main list (and considered by proof steps) in order of their score.
- Algorithm ends when False is derived, or when there are no more items waiting to be processed, or if a timeout condition is reached.

A simple example

- Given an infinite sequence $X = (X_0, X_1, X_2, \ldots)$, assume X is monotone increasing, show $-X$ is monotone decreasing, where $-X$ is defined by $(-X)_i = -(X_i)$.

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- In Isabelle:

$$\text{monotone\_incr}\ X \implies \text{monotone\_decr}\ (-X).$$

- In contradiction form:

$$[\text{monotone\_incr} X, \neg\text{monotone\_decr}\ (-X)] \implies \text{False}.$$

A simple example

monotone_incr X

$\neg$monotone_decr $(-X)$

A simple example

$$\text{monotone\_incr } X$$
$$\downarrow$$
$$\forall m\, n.\; m \le n \longrightarrow X_m \le X_n$$

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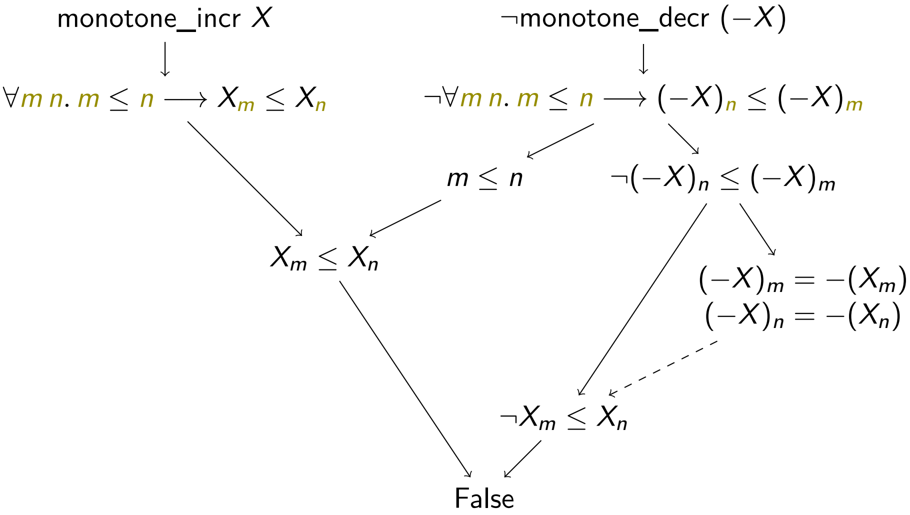

A simple example: linearized trace

1. monotone_incr X (assumption)
2. $\neg$monotone_decr $(-X)$ (assumption)
3. $\forall m\, n.\ m \leq n \longrightarrow X_m \leq X_n$ (1, def. of monotone_incr)
4. $\neg\forall m\, n.\ m \leq n \longrightarrow (-X)_n \leq (-X)_m$ (2, def. of monotone_decr)
5. $m \leq n, \neg(-X)_n \leq (-X)_m$ (4, skolemization)
6. $X_m \leq X_n$ (3 and 5a, quantifier instantiation)
7. $(-X)_m = -(X_m), (-X)_n = -(X_n)$ (5b, def. of $-X$)
8. $\neg X_m \leq X_n$ (5b and 7, inequalities)
9. False (6 and 8, contradiction)

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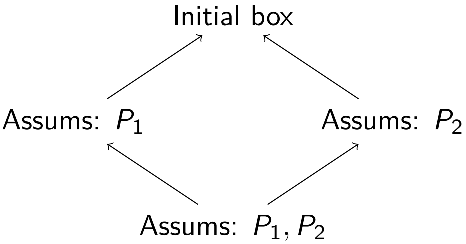
Details: case analysis

- A **box** corresponds to a subcase of the problem. They are specified by a list of additional assumptions. The boxes are organized into a lattice:



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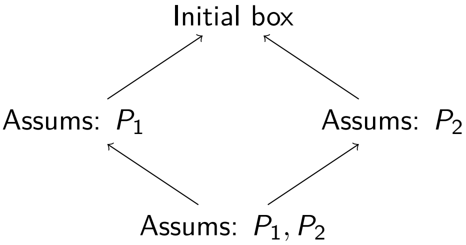

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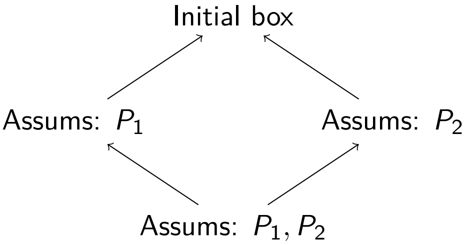

- New boxes are created by proof steps.
- Each item is placed in a box. Placing item P into box with additional assumptions P_1, P_2 is the same as deriving fact $[A_1, \ldots, A_n, \neg C, P_1, P_2] \implies P$.

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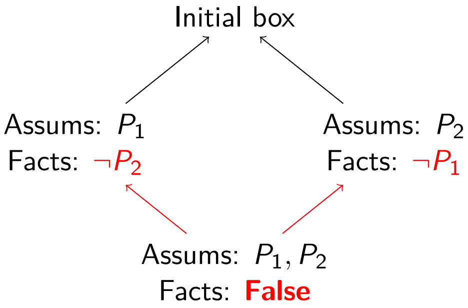

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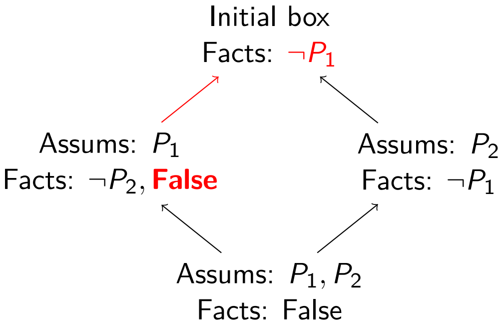
- On resolving the **left** box:



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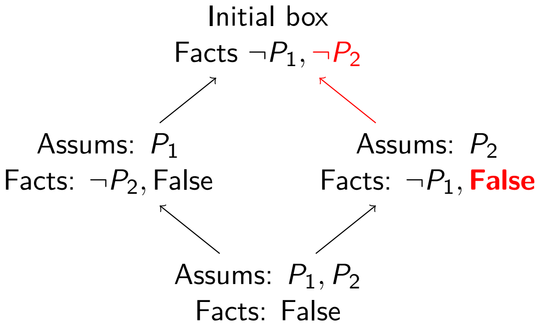
- On resolving the **right** box:



Details: case analysis

- This is similar to how case splitting is handled in the DPLL algorithm, but there are a few differences:
  - Case splits are generated by proof steps, which produce them based on what facts are currently derived.
  - Case splits are not necessarily in sequential order.
  - Derivation in different subcases proceed in parallel.

Details: E-matching

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  - $S = \{y = f(x), z = g(y)\}, p = g(f(?a)), t = z$.

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- Widely used for quantifier instantiation in SMT solvers.

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- Example: given a previously proved theorem of form $[A, B] \implies C$, where $vars(C) \subseteq vars(A) \cup vars(B)$, can write proof step that
  - Perform E-matching on two facts against patterns A and B.
  - For each match, output the instantiated C.

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- Simple proof steps can be added in one line of code (for example, apply a theorem $[A, B] \implies C$ in the forward direction).
- However, arbitrarily complex proof steps can be written in ML, with soundness guaranteed by the LCF architecture (similar to tactics in Isabelle).

Details: Proof scripts

- For a more difficult theorem, users can supply intermediate steps. `auto2` then tries to fill in the gaps between the steps.
  - `OBTAIN` P: prove P, then add P to the set of derived items.
  - `CASE` P: prove a contradiction from P, then add $\neg P$ to the set of derived items.
  - `CHOOSE` $x, P(x)$: prove $\exists x.P(x)$, then obtain new variable x satisfying $P(x)$.
  - C_1 `THEN` C_2: perform C_1, then perform C_2 after C_1 is finished.
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- Similar to Isar, but with simpler structure, and no need to reference names of previous lemmas or tactics.

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Case studies

- Formalizations performed using `auto2`:
  - Elementary theory of prime numbers, up to infinitude of prime numbers and the unique factorization theorem.
  - Functional data structures, including red-black trees.
  - Parts of Hoare logic.
  - Verification of imperative programs (based on Imperative/HOL, without using Hoare or separation logic).
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  - Construction of real numbers using Cauchy sequences.
  - Arrow's impossibility theorem.
- In all case studies, we aim to use `auto2` to prove all major theorems, using proof scripts at a level of detail comparable to usual mathematical exposition.

Case studies: prime numbers

- Lemma `larger_prime` (for proving infinitude of prime numbers):

  $\exists p.\, \mathrm{prime}\ p \wedge n < p$

  with proof script

  CHOOSE p, prime $p \wedge p$ dvd fact $n + 1$ THEN
  CASE $p \leq n$ WITH OBTAIN p dvd fact n

Case studies: prime numbers

- Lemma `factorization_unique_aux` (for proving uniqueness of factorization):

$$\forall p \in \text{set } M.\ \text{prime } p \implies \forall p \in \text{set } N.\ \text{prime } p \implies \prod_{i \in M} i \ \texttt{dvd} \ \prod_{i \in N} i \implies M \subseteq N$$

with proof script

CASE $M = \emptyset$ THEN
CHOOSE $M', m, M = M' + \{m\}$ THEN
OBTAIN $m \ \texttt{dvd} \ \prod_{i \in N} i$ THEN
CHOOSE $n, n \in N \wedge m \ \texttt{dvd} \ n$ THEN
CHOOSE $N', N = N' + \{n\}$ THEN
OBTAIN $m = n$ THEN
OBTAIN $\prod_{i \in M'} i \ \texttt{dvd} \ \prod_{i \in N'} i$ THEN
STRONG_INDUCT $(M, [\text{Arbitrary } N])$

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- Data structures and algorithms verified:
  - Arrays: reverse, quicksort.
  - Linked list: insert, remove, reverse, merge.
  - Binary trees: insert, delete-min.
- Most proofs are either automatic or only need specifying the induction scheme, whereas corresponding proofs using tactics can run for several dozen lines. The theorems also appear to be beyond the reach of Sledgehammer tools.

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- Important result in social choice theory: it is impossible to design a voting system for more than two candidates that satisfy a set of reasonable conditions.
- Proofs of all major lemmas / theorems are done using `auto2`, with slightly fewer subgoals than in the tactics version.

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- For formalization of mathematics: systems of heuristics for real and complex analysis, abstract algebra, number theory, discrete mathematics, etc.
- For verification of imperative programs: incorporating separation logic, as well as further techniques such as symbolic execution and shape analysis.

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- Link to code:

  `https://github.com/bzhan/auto2`