An Isabelle/HOL Formalisation of Green's Theorem

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Abstract

- We formalised a statement of Green's theorem in Isabelle/HOL
- Outline
  - What is Green's theorem?
  - Traditional statement and proof of Green's theorem
  - Our statement and proof of Green's theorem

Stokes' Theorems

- A family of theorems relating functions to the integrals of their derivatives
- 1 dimension: Fundamental Theorem of Calculus, for $f : \mathbb{R} \Rightarrow \mathbb{R}$

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

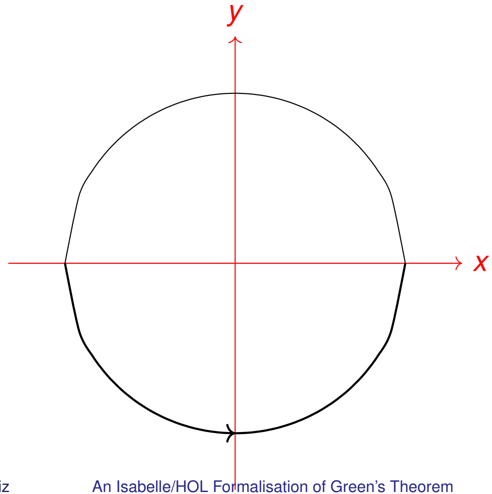
- 2 dimension: Green's Theorem for a field $F : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

Green's Theorem

- a region $D : \mathbb{R}^2$ `set`: satisfying some conditions

$$D = \{(x, y) \mid x^2 + y^2 \leq C\}$$

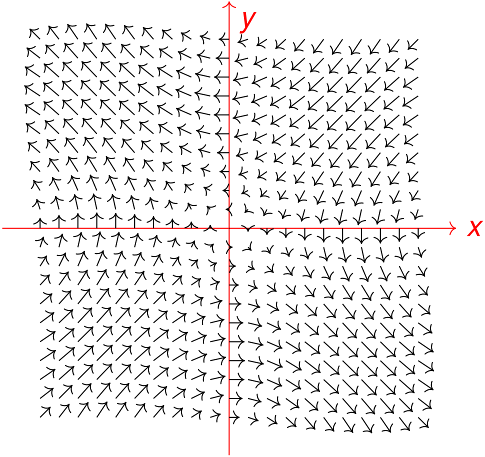


Green's Theorem

- a field $F : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$: satisfying some conditions in and around D

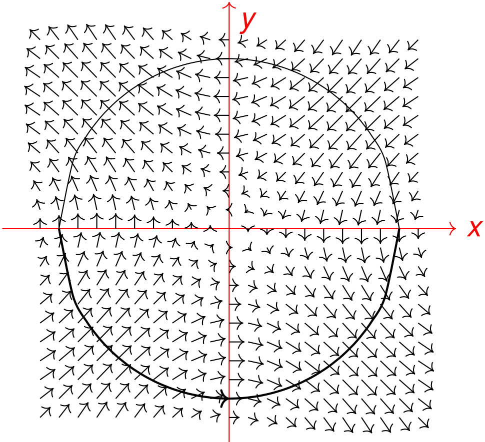
$$F(x, y) = (F_x(x, y), F_y(x, y))$$
$$F_x(x, y) = y^3 - 9y$$
$$F_y(x, y) = x^3 - 9x$$

Green's Theorem



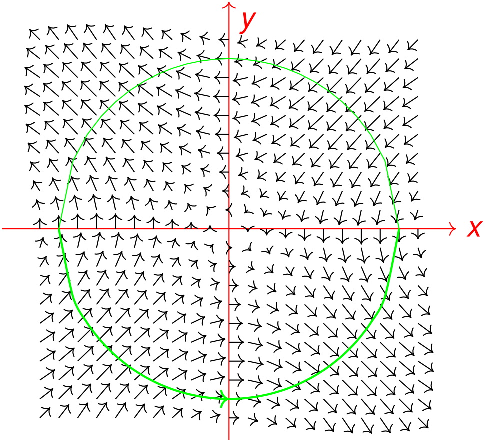
Green's Theorem

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$



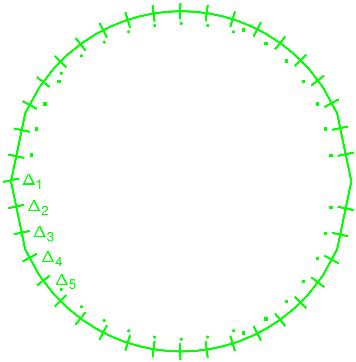
Line integral

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$



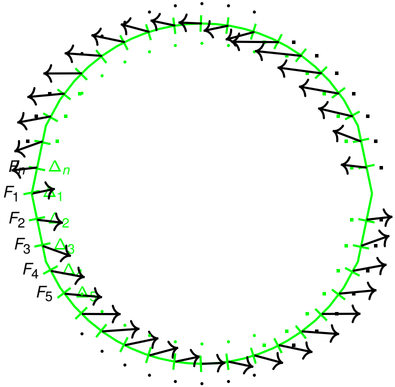
Line integral

$$\Delta_i = (x_{i+1} - x_{i-1}, y_{i+1} - y_{i-1})$$



Line integral

$$\sum_{1}^{n} F_i \bullet \Delta_i = \sum_{1}^{n} F_{x_i}\Delta_{x_i} + F_{y_i}\Delta_{y_i}$$



Line integral

$$\sum_{1}^{n} F_i \bullet \Delta_i = \sum_{1}^{n} F_{x_i} \Delta_{x_i} + F_{y_i} \Delta_{y_i}$$

This summation approximates:

- Rotation of a field
- Circulation of a fluid w.r.t. a boundary
- work done by a field on a particle

Line integral

When $\Delta_i \to 0$ (equiv. $n \to \infty$)

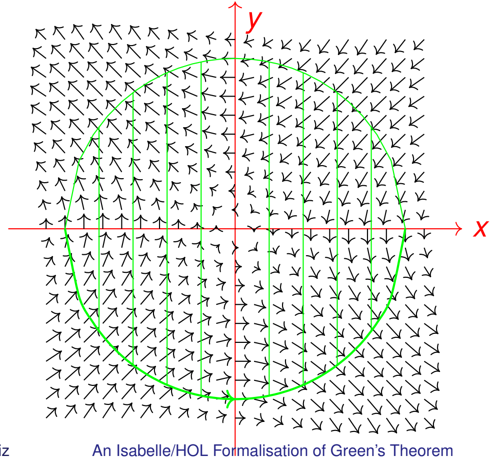
$$\sum_{1}^{n} F_{x_i}\Delta_{x_i} + F_{y_i}\Delta_{y_i} = \int_{\gamma} F \equiv \int_{0}^{1} F_x(\gamma(t))\gamma'_x(t) + F_y(\gamma(t))\gamma'_y(t)dt$$

where the line is parametrised as $\gamma : [0, 1] \Rightarrow \mathbb{R}^2$ (i.e. 1-cube)
In Isabelle/HOL, we defined it on top of Henstock-Kurzweil integral for:

- a field F : Euclidean Space $\Rightarrow$ Euclidean Space.
- projected on a subset of the Basis of the space

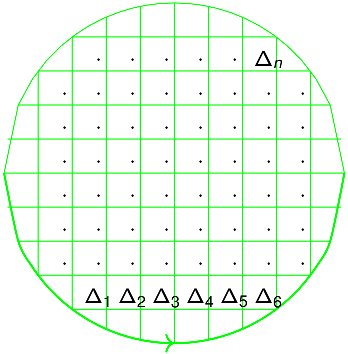
Double integral

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$



Double integral

$$\Delta_i = (x_{i+1} - x_i)(y_{i+1} - y_i)$$



Double integral

The double integral can be approximated by the summation

$$\sum_{i=1}^{n} g(x_i, y_i)\Delta_i$$

where $g : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is a "scalar" function.

- We used the Henstock-Kurzweil integral in Isabelle/HOL to model when $n \to \infty$
- In our case $g = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$, i.e.
  - the rate of change of the line integral w.r.t. area of D
  - models the vorticity of a fluid, or field rotation density, etc.

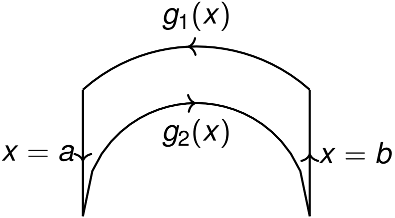
Green's Theorem: Applications

- in mathematical analysis, e.g.
  - derive Cauchy's integral theorem
  - manipulating partial differential equations
- in analytical/mathematical physics, e.g.
  - electromagnetism and electrodynamics: e.g. deriving Faraday's law (point form)
  - astronomy: e.g. deriving Kepler's law for heavenly bodies
- Justification of efficient numerical methods for
  - approximating integral on the boundary $O(n)$ vs $O(n^2)$
  - fluid dynamics
  - image processing

Green's Theorem: Elementary Regions

D_x is a type I region iff there are C^1 smooth functions g_1 and g_2 such that for two constants a and b:

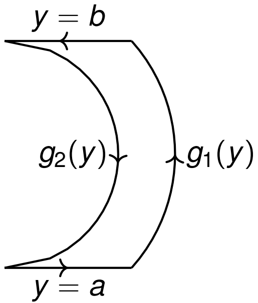
$$D_x = \{(x, y) \mid a \leq x \leq b \wedge g_2(x) \leq y \leq g_1(x)\}$$



Green's Theorem: Elementary Regions

D_y is a type II region iff there are C^1 smooth functions g_1 and g_2 such that for two constants a and b:

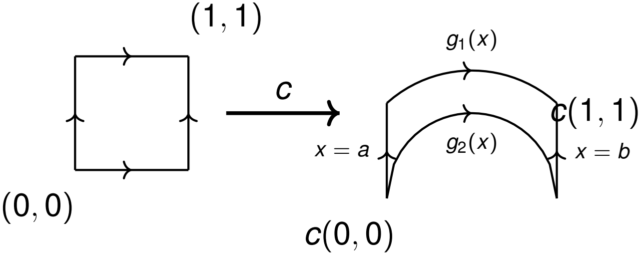
$$D_y = \{(x, y) \mid a \leq y \leq b \wedge g_2(y) \leq x \leq g_1(y)\}$$



Green's Theorem: Elementary Regions

- D_x is formalised as $c : [0,1]^2 \Rightarrow \mathbb{R}^2$ (i.e. 2-cube), such that:

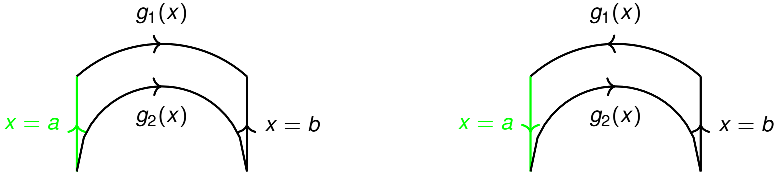
$$c(\![0,1]^2]\!) = \{(x,y) \mid a \leq x \leq b \wedge g_2(x) \leq y \leq g_1(x)\}$$



Green's Theorem: Elementary Regions

- ∂D_x is the set of *oriented* paths (i.e. 1-chain)

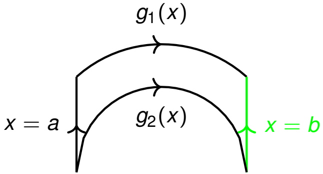
$$\{(-1, (\lambda t.c(0, t))), (1, (\lambda t.c(1, t))), (1, (\lambda t.c(t, 0))), (-1, (\lambda t.c(t, 1)))\}$$

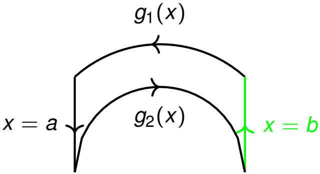


Green's Theorem: Elementary Regions

- ∂D_x is the set of *oriented* paths (i.e. 1-chain)

$$\{(-1,(\lambda t.c(0,t))),(1,(\lambda t.c(1,t))),(1,(\lambda t.c(t,0))),(-1,(\lambda t.c(t,1)))\}$$





Green's Theorem: Elementary Regions

$$\oint_{\partial D_x} F_x dx + F_y dy = \int_{D_x} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

Using

- line integral of F_x on a vertical line is 0
- Fubini's theorem
- algebraic manipulation

Where

- the line integral is lifted to 1-chains

Green's Theorem: Elementary Regions

$$\oint_{\partial D_y} F_x dx + F_y dy = \int_{D_y} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

Using

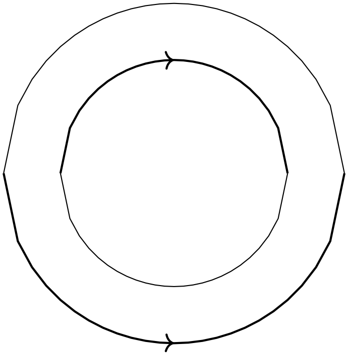
- line integral of F_y on a vertical line is 0
- Fubini's theorem
- algebraic manipulation

Where

- the line integral is lifted to 1-chains

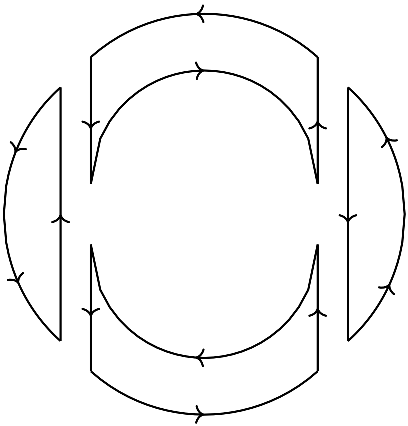
Green's Theorem: Regions with piecewise smooth boundaries

For a region D that can be divided in finitely many Type I 2-cubes (i.e. a type I 2-chain)



Green's Theorem: Regions with piecewise smooth boundaries

For a region D that can be divided in finitely many Type I regions (i.e. a type I 2-chain)



Green's Theorem: Regions with piecewise smooth boundaries

For a region D that can be divided in finitely many Type I regions C_x (i.e. a type I 2-chain)

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

Proof:

- $\oint_{\partial D} F_x dx = \sum_{D_x \in C_x} \oint_{\partial D_x} F_x dx$
- $\sum_{D_x \in C_x} \int_{D_x} - \frac{\partial F_x}{\partial y} \, dxdy = \int_D - \frac{\partial F_x}{\partial y} \, dxdy$
- Half Green's theorem for Type I regions

Green's Theorem: Regions with piecewise smooth boundaries

Similarly, for a region D that can be divided in finitely many Type II regions (i.e. a type II 2-chain)

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

And accordingly we have:

$$\oint_{\partial D} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

if D can be represented into *both* a set of type I regions *and* a set of type II regions.

Green's Theorem: Regions with piecewise smooth boundaries

If D can be represented by *both* a type I 2-chain *and* a type II 2-chain.

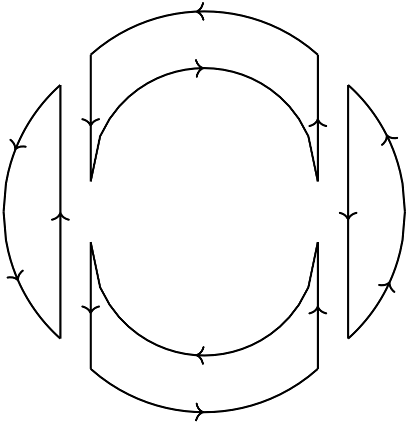
$$\oint_{\partial D} F_x dx + F_y dy = \int_{D} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dxdy$$

Difficulties of formalising this proof

- complex/tedious topological argument of line integral cancellation
- requires formalising paths and their orientations w.r.t. exterior normal

Green's Theorem: Our approach

If D can be divided into a type I 2-chain C_x by adding *only* vertical lines



Green's Theorem: Our approach

If D can be divided into a type I 2-chain C_x by adding *only* vertical lines
We have

$$\int_{\gamma_x} F_x dx = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

for any set of oriented paths γ_x (i.e. 1-chain) that includes *all* the horizontal edges of C_x

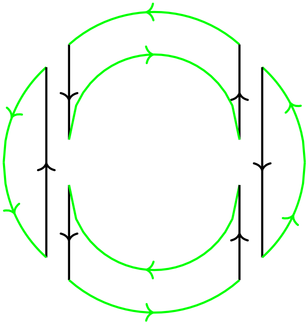
- Because the integral of F_x on any vertical line is zero

Green's Theorem: Our approach

If D can be divided into a type I 2-chain C_x by adding *only* vertical lines

$$\int_{\gamma_x} F_x dx = \int_D -\frac{\partial F_x}{\partial y} \, dxdy$$

γ_x can be



Green's Theorem: Our approach

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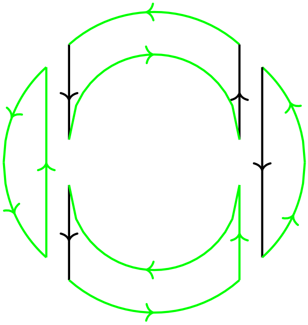

Green's Theorem: Our approach

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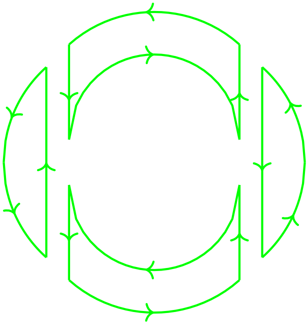


Green's Theorem: Our Approach: Path Equivalence

Similarly, if D can be divided into a type II 2-chain C_y *only* with horizontal lines

$$\int_{\gamma_y} F_y dy = \int_D \frac{\partial F_y}{\partial x}\, dxdy$$

- For any 1-chain γ_y that includes *all* the vertical boundaries of C_y
- Because the integral of F_y on any horizontal line is zero

Green's Theorem: Our Approach: Path Equivalence

We have

$$\int_{\gamma_x} F_x dy = \int_D -\frac{\partial F_x}{\partial y}\, dxdy$$

and

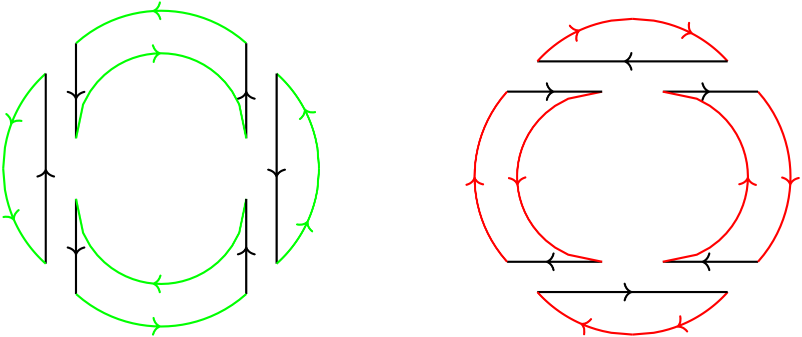
$$\int_{\gamma_y} F_y dy = \int_D \frac{\partial F_y}{\partial x}\, dxdy$$

How can we combine them?

- It is not straight-forward because γ_x and γ_y are not necessarily the same

Green's Theorem: Our Approach: Path Equivalence

Example γ_x, and γ_y



They are equivalent, but not the same

Green's Theorem: Our Approach: Path Equivalence

Their equivalence can be captured by

- formalising paths and their orientations w.r.t. exterior normal, OR
- the concept of a *common subdivision*

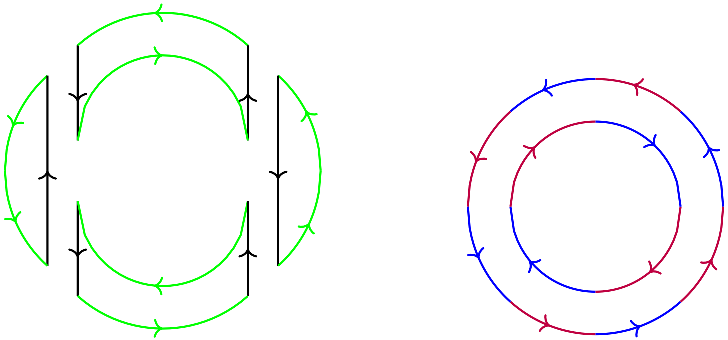
Green's Theorem: Our Approach: Path Equivalence

1-chain γ_1 is a subdivision of 1-chain γ_2 iff

- for every path (i.e. 1-cube) $c \in \gamma_2$,
  - there is a list ordering of a subset of γ_1 that subdivides c
- One way of capturing the equivalence of two 1-chains is the existence of a common subdivision

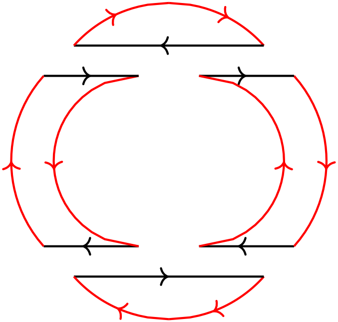
Green's Theorem: Our Approach: Path Equivalence

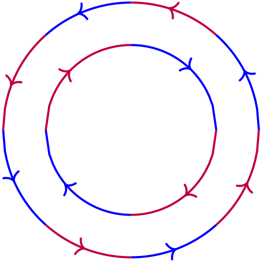
A subdivision of γ_x



Green's Theorem: Our Approach: Path Equivalence

A subdivision of γ_y





Green's Theorem: Our Approach: Path Equivalence

1-chain γ_1 is a subdivision of 1-chain γ_2 iff

- for every cube $c \in \gamma_2$,
  - there is a subset of γ_1 with a list ordering that subdivides c
- One way of capturing the equivalence of two 1-chains is the existence of a common subdivision

Lemma

For 1-chains γ_1 and γ_2, if there is a common subdivision between them, then

$$\int_{\gamma_1} F_x dx + F_y dy = \int_{\gamma_2} F_x d_x + F_y dy$$

Green's Theorem: Our Approach: Path Equivalence

Theorem (Green's Theorem)

If D can be represented by both a type I 2-chain C_x and a type II 2-chain C_y

- *using only vertical and horizontal lines, respectively.*

for any 1-chain γ_x that that includes all the horizontal edges of C_x

$$\oint_{\gamma_x} F_x dx + F_y dy = \int_D \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dxdy$$

Green's Theorem: Our Approach: Generality: Geometrical Assumptions

Conjecture

If D can be be represented both by type I 2-chain and type II 2-chain, then

- *D can be represented by both a type I 2-chain C_x and a type II 2-chain C_y*
  - *using only vertical and horizontal lines, respectively.*

Green's Theorem: Our Approach: Generality: Analytic Assumptions

Our theorem's analytic assumptions are

- F_x and F_y are continuous in D
- $\frac{\partial F_x}{\partial y}$ and $\frac{\partial F_x}{\partial y}$ exist and are Lebesgue integrable in D

More general than the assumption, commonly used in analysis books

- F and all of its partial derivatives are continuous in D

Green's Theorem: Proof Practicalities

- Previous formalisations that we used:
  - the Probability and the multivariate analysis libraries from Isabelle/HOL
  - Paulson's porting of Harrison's HOL light complex analysis
- Size of the formalisation 7.5K lines
- Around 3 months of work to learn Isabelle and formalise the theorem

Green's Theorem: Conclusions and Future Work

- We formalised a sufficiently general statement of Green's theorem
- This was facilitated by a new argument
- As future work:
  - generalise this argument to prove the general Stokes' theorem
  - will at least need a multivariate change of variable theorem